Higher-Loop Structural Properties of the β Function in Asymptotically Free Vectorial Gauge Theories

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We investigate some higher-loop structural properties of the β function in asymptotically free vectorial gauge theories. Our main focus is on theories with fermion contents that lead to an infrared (IR) zero in β. We present analytic and numerical calculations of the value of the gauge coupling where β reaches a minimum, the value of β at this minimum, and the slope of β at the IR zero, at two-, three-, and four-loop order. The slope of β at the IR zero is relevant for estimates of a dilaton mass in quasiconformal gauge theories. Some inequalities are derived concerning the dependence of the above quantities on loop order. A general inequality is derived concerning the dependence of the shift of the IR zero of β, from the n-loop to the $(n+1)$-loop order, on the sign of the $(n+1)$-loop coefficient in β. Some results are also given for gauge theories with $\mathcal{N}=1$ supersymmetry.

I. INTRODUCTION

The evolution of an asymptotically free gauge theory from high Euclidean momentum scales μ in the deep ultraviolet (UV) to small scales in the infrared is of fundamental field-theoretic interest. This evolution is described by the β function of the theory. Following the pioneering calculations of the β function at one-loop [1] and two-loop [2] order, this function was subsequently calculated to three-loop [3] and four-loop [4] order in the modified minimal [5] subtraction ($\overline{MS}$) scheme [6]. The anomalous dimension of the (gauge-invariant) fermion bilinear operator, γ_m, has also been calculated up to four-loop order in this scheme [7].

Here we consider the UV to IR evolution of an asymptotically free vectorial gauge theory with gauge group G and N_f massless fermions transforming according to a representation R of G [8]. An interesting property of this type of theory is that, for sufficiently large N_f, the two-loop β function has an IR zero [2, 9]. If N_f is near to the maximum allowed by the property of asymptotic freedom, then this IR zero occurs at a small value, but, as N_f decreases, it increases to stronger coupling. This motivates the calculation of the IR zero of β at higher-loop order [10]. Calculations of this IR zero, and the associated anomalous dimension of the (gauge-invariant) fermion bilinear, γ_m, have recently been done to four-loop order for an asymptotically free vectorial gauge theory with gauge group G and N_f fermions in an arbitrary representation R, with explicit results for R equal to the fundamental, adjoint, and symmetric and antisymmetric rank-2 tensor representations [11, 12]. A corresponding analysis was carried out for an asymptotically free vectorial gauge theory with $\mathcal{N}=1$ supersymmetry in [13]. Although the terms in the β function at three- and higher-loop order, and the terms in γ_m at two- and higher-loop order are dependent on the scheme used for regularization and renormalization of the theory, these higher-loop calculations are valuable because they give a quantitative measure of the accuracy and stability of the lowest-order calculations of α_{IR} and γ_m. A study of the effect of scheme transformations on results for α_{IR} was performed in [14].

In this paper we will present calculations at the n-loop level, where $n = 2, 3, 4$, of several important quantities that provide a detailed description of the UV to IR evolution of a theory with an IR zero in its β function. Our general results apply for an arbitrary (non-Abelian) gauge group G. We denote the running gauge coupling at a scale μ as $g(\mu)$, and define $\alpha(\mu) = g(\mu)^2/(4\pi)$. (The μ argument will often be suppressed in the notation.) The loop order to which a quantity is calculated is indicated explicitly via the subscript $n\ell$, standing for n-loop, so that the n-loop β function and its IR zero are denoted $\beta_{n\ell}$ and $\alpha_{IR,n\ell}$. Given the asymptotic freedom of the theory, the UV to IR evolution, as described by $\beta_{n\ell}$, occurs in the interval

$$I_\alpha : \quad 0 \le \alpha(\mu) \le \alpha_{IR,n\ell} \ . \tag{1.1}$$

In addition to $\alpha_{IR,n\ell}$, the three structural properties of $\beta_{n\ell}$ that we study are (i) the value of α where $\beta_{n\ell}$ reaches its minimum in the interval (1.1), denoted $\alpha_{m,n\ell}$, (ii) the minimum value of $\beta_{n\ell}$ on this interval, $(\beta_{n\ell})_{min}$, and (iii) the slope of $\beta_{n\ell}$ at $\alpha_{IR,n\ell}$, denoted $d\beta_{n\ell}/d\alpha|_{\alpha_{IR,n\ell}}$. The importance of the first two quantities for the UV to IR evolution of the theory is clear. One would like to know where the rate of running, $\beta = d\alpha/dt$, has maximum magnitude, as a function of α, and hence, as a function of μ. Further, one is interested in what this maximum magnitude in the rate of running, i.e., (since $\beta \le 0$), the minimum value of β is in the interval I_α. The third quantity, the slope of the β function at α_{IR}, is of interest because it describes how rapidly β approaches zero as α approaches α_{IR}. A knowledge of this slope is also valuable because it is relevant for estimates of

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a dilaton mass in gauge theories that exhibit approximate scale-invariance associated with an IR zero of β at a value, α_{IR}, that is sufficiently large that this approximate dilatation symmetry is broken by the formation of a fermion condensate. For each of these structural quantities, one would like to see how higher-loop calculations compare with the two-loop computation. As part of our work, we derive some inequalities concerning the relative values of each of these quantities at the two- and three-loop order.

We also generalize some results that were obtained in [11] concerning $\alpha_{IR,n\ell}$. In [11] it was shown that in a theory with a given G, R, and N_f for which the two-loop β function $\beta_{2\ell}$ has an IR zero, the three-loop zero satisfies the inequality $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ in the minimal subtraction ($\overline{MS}$) scheme used there [15]. The reduction in the value of the IR zero going from two-loop to three-loop level is typically substantial; for example, for $G = \mathrm{SU}(2)$ and $N_f = 8$, $\alpha_{IR,2\ell} = 1.26$, while $\alpha_{IR,3\ell} = 0.688$, and for $G = \mathrm{SU}(3)$ and $N_f = 12$, $\alpha_{IR,2\ell} = 0.754$, while $\alpha_{IR,3\ell} = 0.435$. A natural question that arises from the analysis in [11] is how general this inequality is and, specifically, whether it also holds for other schemes. We address and answer this question here. We prove that for an asymptotically free theory with a given G, R, and N_f for which $\beta_{2\ell}$ has an IR zero, the inequality $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$ holds in any scheme which has the property that the sign of the three-loop coefficient in β is opposite to that of the one-loop coefficient for $N_f \in I$, which thus preserves for $\beta_{3\ell}$ the existence of an IR zero that was true of $\beta_{2\ell}$. This preservation of the two-loop IR zero in β is physically desirable, since $\beta_{2\ell}$ is scheme-independent, so if it exhibits an IR zero, then a reasonable scheme should maintain the existence of this zero at higher-loop level. More generally, we will derive a result that shows how $\alpha_{IR,n\ell}$ shifts, upward or downward, to $\alpha_{IR,(n+1)\ell}$, when it is calculated to the next higher-loop order.

For a given gauge group G, the infrared properties of the theory depend on the fermion representation R and the number of fermions, N_f. For a sufficiently large number, N_f, of fermions in a given representation (as bounded above by the requirement of asymptotic freedom), the IR zero in β occurs at a relatively small value of α and the theory evolves from the UV to the IR without any spontaneous chiral symmetry breaking (SχSB). In this case, the IR zero of β is an exact infrared fixed point of the renormalization group. Thus, the infrared behavior of the theory exhibits scale-invariance (actually conformal invariance [16]) in a non-Abelian Coulomb phase. For small N_f, as the theory evolves from the UV to the IR, and the reference scale μ decreases below a scale which may be denoted Λ, the gauge interaction becomes strong enough to confine and produce bilinear fermion condensates, with the associated spontaneous chiral symmetry breaking and dynamical generation of fermion masses of order Λ. As μ decreases below Λ, and one constructs the effective low-energy field theory applicable in this region, one thus integrates out these now-massive fermions, and the β function changes to that of a pure gauge theory, which does not have any perturbative IR zero. Hence, in this case the infrared zero of β is an approximate, but not exact, fixed point of the renormalization group.

If N_f is only slightly less than the critical value $N_{f,cr}$ for spontaneous chiral symmetry breaking, so that α_{IR} is only slightly greater than the critical value, α_{cr} (depending on G and R) for fermion condensation, then the UV to IR evolution exhibits approximate scale (dilatation) invariance for an extended logarithmic interval, because as $\alpha(\mu)$ increases toward α_{IR}, while less than α_{cr}, β approaches zero, i.e., the rate of change of $\alpha(\mu)$ as a function of μ approaches zero. Thus, $\alpha(\mu)$ is large, of O(1), but slowly running ("walking"). This is quite different from the behavior of $\alpha_s(\mu)$ in quantum chromodynamics (QCD). This approximate scale invariance at strong coupling plays an important role in models with dynamical electroweak symmetry breaking [17, 18], and occurs naturally in models with an approximate infrared fixed point [18]. Since $\alpha_{IR} \sim O(1)$ and γ_m is a power series in α, there is an enhancement of γ_m in such models, which, in turn, is useful for generating sufficiently large Standard Model fermion masses. Approximate calculations of hadron masses and related quantities have been performed using continuum field theoretic methods for these theories [19]. Recently, an intensive effort has been made using lattice methods to study the properties of $\mathrm{SU}(N_c)$ gauge theories with various fermion contents, in particular, theories that exhibit quasi-scale-invariant behavior associated witn an exact or approximate IR zero of the respective β functions. For example, for SU(3) with fermions in the fundamental representation, measurements of γ_m have been reported in [20]. In theories where N_c, R, and N_f are such that α_{IR} is only slightly greater than α_{cr}, so this approximate scale invariance associated with an IR zero of β at strong coupling holds, the spontaneous breaking of this symmetry by the formation of a fermion condensate, may lead to a light state which is an approximate Nambu-Goldstone boson (NGB), the dilaton [21] (see also [19]). The mass of the dilaton depends on several quantities, including the effective value of the β function at the relevant scale $\mu \sim \Lambda$ where the SχSB takes place. The desire to study the quasi-scale-invariant behavior of such a theory is an important motivation for obtaining more detailed information about the structure of the β function, as contained in the structural quantities (i)-(iii) discussed above.

II. BETA FUNCTION

A. General

The UV to IR evolution of the theory is described by the β function

$$\beta \equiv \beta_\alpha \equiv \frac{d\alpha}{dt} \ , \tag{2.1}$$

where $t = \ln \mu$. This has the series expansion

$$\beta = -2\alpha \sum_{\ell=1}^{\infty} b_\ell \, a^\ell = -2\alpha \sum_{\ell=1}^{\infty} \bar{b}_\ell \, \alpha^\ell \ , \tag{2.2}$$

where ℓ denotes the number of loops involved in the calculation of b_ℓ, $a \equiv g^2/(16\pi^2) = \alpha/(4\pi)$, and

$$\bar{b}_\ell = \frac{b_\ell}{(4\pi)^\ell} \ . \tag{2.3}$$

As noted above, the one- and two-loop coefficients b_1 and b_2, which are scheme-independent, were calculated in [1, 2] (see Appendix A). The b_ℓ for $\ell \ge 3$ are scheme-dependent; in the commonly used $\overline{MS}$ scheme, the b_ℓ have been calculated up to four-loop order [3, 4]. For analytical purposes it is more convenient to deal with the b_ℓ, since they are free of factors of 4π. However, for numerical purposes, it is usually more convenient to use the $\bar{b}_\ell$, since, as is evident from Table I of [11], the range of values of $\bar{b}_\ell$ is smaller than the range for the b_ℓ. We will use both interchangeably. We denote the β function calculated to n-loop order as

$$\beta_{n\ell} = -8\pi \sum_{\ell=1}^{n} b_\ell \, a^{\ell+1} = -2 \sum_{\ell=1}^{n} \bar{b}_\ell \, \alpha^{\ell+1} \ . \tag{2.4}$$

Some explicit examples of four-loop β functions are given in Appendix B.

With our sign conventions, the restriction to an asymptotically free theory means that $b_1 > 0$. This is equivalent to the condition

$$N_f < N_{f,b1z} \ , \tag{2.5}$$

where [22, 23]

$$N_{f,b1z} = \frac{11 C_A}{4 T_f} \ . \tag{2.6}$$

($b\ell z$ stands for b_ℓ zero). For the fundamental, adjoint, and symmetric and antisymmetric rank-2 tensor representations of $G = {\rm SU}(N_c)$, the upper bound (2.5) allows the following ranges of N_f: (i) $N_f < (11/2)N_c$ for fundamental, (ii) $N_f < 11/4$ for adjoint, (iii) $N_f < 11N_c/[2(N_c \pm 2)]$ for symmetric (antisymmetric) rank-2 tensor. In the case of a sufficiently large representation R, this upper bound may forbid even the value $N_f = 1$. For example, for the rank-3 symmetric tensor representation of ${\rm SU}(N_c)$, the upper bound is $N_f < 11N_c/[(N_c + 2)(N_c + 3)]$, and the right-hand side of this bound is larger than 1 only for N_c (analytically continued to nonnegative real numbers) in the interval $3 - \sqrt{3} < N_f < 3 + \sqrt{3}$, i.e., $1.268 < N_f < 4.732$ (to the indicated floating-point accuracy). Hence, if N_c is equal to 2, 3, or 4, the bound allows only the single value $N_f = 1$, and if $N_c \ge 5$, then the bound does not allow any nonzero (integer) value of N_f. For $G = {\rm SU}(2)$, with a representation labeled by the integer or half-integer j, the inequality (2.5) is

$$N_f < \frac{33}{2j(j+1)(2j+1)} \quad \text{for} \quad G = {\rm SU}(2) \ . \tag{2.7}$$

This bound is: (i) $N_f < 11$ if $j = 1/2$; (ii) $N_f < 11/4$ if $j = 1$; (iii) $N_f < 11/10$ if $j = 3/2$. The right-hand side of (2.7) decreases through 1 as j (continued to real numbers) increases through 1.562, so that the upper bound (2.7) does not allow a nonzero number of fermions in a representation of SU(2) with $j \ge 2$ [24].

To analyze the zeros of the n-loop β function, $\beta_{n\ell}$, aside from the double zero at $\alpha = 0$, one extracts the overall factor of $-2\alpha^2$ and calculates the zeros of the reduced (r) polynomial

$$\beta_{n\ell,r} \equiv -\frac{\beta_{n\ell}}{2\alpha^2} = \sum_{\ell=1}^{n} \bar{b}_\ell \, \alpha^{\ell-1} \ . \tag{2.8}$$

or equivalently, $\sum_{\ell=1}^{n} b_\ell \, a^{\ell-1}$. As is clear from Eq. (2.8), the zeros of $\beta_{n\ell}$ away from the origin depend only on $n-1$ ratios of coefficients, which can be taken as $\bar{b}_\ell/\bar{b}_n$ for $\ell = 1, ..., n-1$. Although Eq. (2.8) is an algebraic equation of degree $n-1$, with $n-1$ roots, only one of these is physically relevant as the IR zero of $\beta_{n\ell}$. We denote this as $\alpha_{IR,n\ell}$. In analyzing how the n-loop β function describes the UV to IR evolution of the theory, we will focus on the interval (1.1).

To investigate how $\alpha_{IR,n\ell}$ changes when one calculates it to higher-loop order, it is useful to characterize the full set of zeros of $\beta_{n\ell}$. In general, if one has a polynomial of degree m, $P_m(z) = \sum_{s=0}^{m} \kappa_s z^s$, and one denotes the set of m roots of the equation $P_m(z) = 0$ as $\{z_1, ..., z_m\}$, then the discriminant of this equation is defined as [25]

$$\Delta_m \equiv \Big[\kappa_m^{m-1} \prod_{i<j} (z_i - z_j) \Big]^2 \ . \tag{2.9}$$

Since Δ_m is a symmetric polynomial in the roots of the equation $P_m(z) = 0$, the symmetric function theorem implies that it can be expressed as a polynomial in the coefficients of $P_m(z)$ [26]. We will sometimes indicate this dependence explicitly, writing $\Delta_m(\kappa_0, ..., \kappa_m)$. The discriminant Δ_m is a homogeneous polynomial of degree $m(m-1)$ in the roots $\{z_i\}$. For our present purpose, to analyze the zeros of $\beta_{n\ell}$ away from the origin, given by the roots of Eq. (2.8), of degree $m = n-1$, we will thus use the discriminant $\Delta_{n-1}(\bar{b}_1, \bar{b}_2, ..., \bar{b}_n)$, or equivalently,

$\Delta_{n-1}(b_1, b_2, ..., b_n)$. Note that, because of the homogeneity properties,

$$\Delta_{n-1}(\bar{b}_1, \bar{b}_2, ..., \bar{b}_n) = (4\pi)^{-(n+1)(n-2)} \Delta_{n-1}(b_1, b_2, ..., b_n) . \tag{2.10}$$

Some further details on discriminants are given in Appendix C.

Although we focus on the behavior of $\beta_{n\ell}$ in the physical interval (1.1), in characterizing the zeros of $\beta_{n\ell}$, we will make use of some formal mathematical properties of $\beta_{n\ell}$ as an abstract function of α. For large $|\alpha|$, since $\beta_{n\ell} \sim -2\bar{b}_n \alpha^{n+1}$, it follows that if α is large and positive, then $\text{sgn}(\beta_{n\ell}) = -\text{sgn}(b_n)$, while for large negative α, $\text{sgn}(\beta_{n\ell}) = \text{sgn}((-1)^n b_n)$. Thus, $\text{sgn}(\beta_{n\ell})$ for large positive α is equal to $(-1)^{n+1}\text{sgn}(\beta_{n\ell})$ for large negative α. Since $\beta_{n\ell}$ is negative in the vicinity of the origin, it follows that for (both even and odd) $n \ge 2$,

$$\text{If } b_n < 0, \text{ then } \beta_{n\ell} \text{ has at least one zero at a positive real value of } \alpha . \tag{2.11}$$

Furthermore, again because $\beta \sim -2\bar{b}_n \alpha^{n+1}$ for large $|\alpha|$, a consequence is that for $n \ge 2$,

$$\text{If } n \text{ is odd and } b_n < 0 \text{ or } n \text{ is even and } b_n > 0, \text{ then } \beta_{n\ell} \text{ has at least one zero at a negative real value of } \alpha . \tag{2.12}$$

Of course, the behavior of $\beta_{n\ell}$ at negative values of α is not directly physical, and the behavior at large positive α is beyond the range of validity of the perturbative calculation, but these mathematical properties will be useful in characterizing the total set of zeros of $\beta_{n\ell}$ at higher-loop order.

Given that $N_f \in I$, so that $\beta_{2\ell}$ has an IR zero, we can track how this zero changes as the loop order n increases. One general result is as follows. As $N_f \nearrow N_{f,b1z}$ at the upper end of the interval I, $\alpha_{IR,n\ell} \to 0$. This is a result of the fact that in this limit, $\bar{b}_1 \to 0$, so that $\beta_{n\ell,r}$ reduces to $\alpha \sum_{\ell=2}^n \alpha^{\ell-2}$, which has a root at $\alpha = 0$. Starting at the ($n = 2$)-loop level and tracking the physical IR zero at three- and higher-loop order, one can infer that generically $\alpha_{n,\ell}$ is the root of $\beta_{n\ell,r}$ that moves toward zero in this limit $N_f \nearrow N_{f,b1z}$.

Because $\beta_{n\ell}$ is a polynomial in $\alpha(\mu)$ and hence a continuous function, and because $\beta_{n\ell} = 0$ at the two ends of the interval (1.1), at $\alpha = 0$ and $\alpha = \alpha_{IR,n\ell}$, and is negative for small (positive) α, it follows that β reaches a minimum in this interval (1.1). This occurs at a point where $d\beta_{n\ell}/d\alpha = 0$, which we label $\alpha_{m,n\ell}$ (where the subscript m stands for "minimum β in I_α"), and we denote

$$(\beta_{n\ell})_{min} \equiv \beta_{n\ell}\Big|_{\alpha=\alpha_{m,n\ell}} . \tag{2.13}$$

From Eq. (2.4), one calculates $d\beta_{n\ell}/d\alpha = (4\pi)^{-1} d\beta_{n\ell}/da$, with the result

$$\frac{d\beta_{n\ell}}{d\alpha} = -2\sum_{\ell=1}^{n} (\ell+1) b_\ell \, a^\ell = -2\sum_{\ell=1}^{n} (\ell+1) \bar{b}_\ell \, \alpha^\ell . \tag{2.14}$$

The equation for the critical points, where $d\beta_{n\ell}/d\alpha = 0$, is thus an algebraic equation of degree n, with n formal roots, one of which is $\alpha = 0$. Assuming that $b_2 < 0$, i.e., $N_f \in I$, so that the two-loop β function has an IR zero (and also, in higher-loop calculations, that the scheme preserves the existence of this IR zero), it follows that, among the remaining $n-1$ roots, one is real and positive and yields the minimum value of $\beta_{n\ell}$ for α in the relevant interval (1.1), and this root is the above-mentioned $\alpha_{m,n\ell}$.

Given that β has an IR zero at α_{IR} and is analytic at this point, one may expand it in a Taylor series about α_{IR}. This involves the slope of the β function at α_{IR}. For compact notation, we denote

$$\beta'_{IR} \equiv \frac{d\beta}{d\alpha}\Big|_{\alpha=\alpha_{IR}} \tag{2.15}$$

and, for the n-loop quantities,

$$\beta'_{IR,n\ell} \equiv \frac{d\beta_{n\ell}}{d\alpha}\Big|_{\alpha=\alpha_{IR,n\ell}} . \tag{2.16}$$

With $\beta(\alpha_{IR}) = 0$, the expansion of $\beta(\alpha)$ for α near to α_{IR} is

$$\beta = \beta'_{IR}\,(\alpha - \alpha_{IR}) + O\Big((\alpha - \alpha_{IR})^2\Big) . \tag{2.17}$$

Here we have written this expansion for the full β function; a corresponding equation applies for $\beta_{n\ell}$.

B. IR Zero of β at the Two-Loop Level

We next review some background on the two-loop β function that is relevant for present work. The two-loop β function is $\beta_{2\ell} = -2\alpha^2(\bar{b}_1 + \bar{b}_2\alpha)$. This has an IR zero at

$$\alpha_{IR,2\ell} = -\frac{\bar{b}_1}{\bar{b}_2} = -\frac{4\pi b_1}{b_2} , \tag{2.18}$$

which is physical if and only if $b_2 < 0$. The coefficient b_2 is a linear, monotonically decreasing function of N_f, which is positive for zero and small N_f and passes through zero, reversing sign, as N_f increases through $N_{f,b2z}$, where

$$N_{f,b2z} = \frac{17C_A^2}{2T_f(5C_A + 3C_f)} . \tag{2.19}$$

For arbitrary G and R, $N_{f,b1z} > N_{f,b2z}$, as is proved by the fact that

$$N_{f,b1z} - N_{f,b2z} = \frac{3C_A(7C_A + 11C_f)}{4T_f(5C_A + 3C_f)} > 0 . \tag{2.20}$$

Hence, there is always an interval I of N_f values for which the two-loop β function has an IR zero, namely

$$I: \quad N_{f,b2z} < N_f < N_{f,b1z} \ . \tag{2.21}$$

For example, in the case of fermions in the fundamental representation, denoted $\Box$,

$$I: \quad \frac{34N_c^3}{13N_c^2-3} < N_f < \frac{11N_c}{2} \quad \text{if } R=\Box \ , \tag{2.22}$$

so that, for $N_c = 2$, the interval I is $5.55 < N_f < 11$; for $N_c = 3$, I is $8.05 < N_f < 16.5$; and as $N_c \to \infty$, I approaches $34N_c/13 < N_f < 11N_c/2$.

Since we are primarily interested in studying the IR zero of β and since the presence or absence of an IR zero of the two-loop β function, $\beta_{2\ell}$, is a scheme-independent property, we focus on $N_f \in I$, where this IR zero of $\beta_{2\ell}$ is present. A general result is that for a given gauge group G and fermion representation R and $N_f \in I$, $\alpha_{IR,2\ell}$ is a monotonically decreasing function of N_f. As N_f decreases from $N_{f,b1z}$, $\alpha_{IR,2\ell}$ increases from 0. As N_f decreases through a value labeled $N_{f,cr}$, α_{IR} increases through a critical value, $\alpha_{cr} \sim O(1)$, where fermion condensation takes place. Thus,

$$N_f = N_{f,cr} \iff \alpha_{IR} = \alpha_{cr} \ . \tag{2.23}$$

The value of $N_{f,cr}$ is of fundamental importance in the study of a non-Abelian gauge theory, since it separates two different regimes of IR behavior, viz., an IR conformal phase with no SχSB for $N_{f,cr} < N_f$ and an IR phase with SχSB for $N_f < N_{f,cr}$. As N_f approaches $N_{f,b2z}$ at the lower end of the interval I, $\alpha_{IR,2\ell}$ becomes too large for Eq. (2.18) to be reliable.

Because of the strong-coupling nature of the physics at an approximate IR fixed point with $\alpha_{IR} \sim O(1)$, there are significant higher-order corrections to results obtained from the two-loop β function, which motivated the calculation of the location of the IR zero in β, and the resultant value of γ_m evaluated at this IR zero, to higher-loop order for a general G, R, and N_f [11, 12].

C. β Function and Dilaton Mass

Here we focus on a theory in which the IR zero of β, α_{IR}, is slightly greater than α_{cr}, so that, in the UV to IR flow, there is an extended interval in $t = \ln\mu$ over which $\alpha(\mu)$ is approaching α_{IR} from below, but is still less than α_{cr}. In this interval, $\alpha(\mu) \sim O(1)$, but β is small, and hence the theory is approximately scale-invariant. As μ decreases through Λ, $\alpha(\mu)$ increases through $\alpha(\Lambda) = \alpha_{cr}$, the fermion condensate forms, and the fermions gain dynamical masses, this approximate scale invariance is broken spontaneously. In terms of the (symmetric) energy-momentum tensor $\theta^{\mu\nu}$, the dilatation current is $D^\mu = \theta^{\mu\nu}x_\nu$, and one has $\partial_\mu D^\mu = [\beta/(4\alpha)]G^a_{\mu\nu}G^{a\mu\nu}$, where $G^a_{\mu\nu}$ is the field-strength tensor for the gauge field. When taking matrix elements, the deviation of this divergence $\partial_\mu D^\mu$ from zero, i.e., the nonconservation of the dilatation current, thus arises from two sources, namely the facts that β is not exactly equal to zero and the nonzero value of the matrix element of $G^a_{\mu\nu}G^{a\mu\nu}$, defined appropriately at the scale Λ. An analysis of the matrix element of $\mathcal{D}^\mu$ between the vacuum and the dilaton state $|\chi(p)\rangle$, in conjunction with a dimensional estimate of the gluon matrix element, and the Taylor series expansion (2.17) evaluated with $\mu \sim \Lambda$ yields the resulting estimate for the dilaton mass m_χ [21]

$$m_\chi^2 \simeq \beta'_{IR}\,(\alpha_{IR} - \alpha_{cr})\Lambda^2 \ . \tag{2.24}$$

In terms of n-loop level quantities, the right-hand side of Eq. (2.24) $\beta'_{IR,n\ell}\,(\alpha_{IR,n\ell} - \alpha_{cr})\Lambda^2$. The importance of the slope at α_{IR}, β'_{IR}, and the n-loop calculation of this slope, $\beta'_{IR,n\ell}$, in estimating a dilaton mass in a quasiconformal theory is evident from Eq. (2.24). As is the case with $\alpha_{IR,n\ell}$, because of the strong-coupling nature of the physics, it is valuable to compute higher-loop corrections to the two-loop result, $\beta'_{IR,2\ell}$. Below, we will present two- and higher-loop analytic and numerical calculations of $\beta'_{IR,n\ell}$. Other effects on m_χ have been discussed in the literature [21], including the effect of dynamical fermion mass generation associated with the spontaneous chiral symmetry breaking as μ descends through the value Λ. Owing to this and other nonperturbative effects on m_χ, we restrict ourselves here to presenting one input to this calculation, namely $\beta'_{IR,n\ell}$, for which we can give definite analytic and numerical results.

D. IR Zero of β at the Three-Loop Level

Let us assume that $N_f \in I$, so that $\beta_{2\ell}$ has an IR zero. Here we analyze how this IR changes as one calculates the β function to three-loop order, extending our results in [11] to (an infinite set of) schemes more general than the $\overline{MS}$ scheme used in that paper. Since the existence of of the IR zero in the two-loop β function is a scheme-independent property of the theory, it is reasonable to restrict to schemes that preserve this IR zero of β at the three-loop level. We first determine a condition for this to hold.

The three-loop β function is $\beta_{3\ell} = -2\alpha^2\beta_{3\ell,r}$, so, aside from the double zero at $\alpha = 0$ (the UV fixed point), $\beta_{3\ell}$ vanishes at the two roots of the factor $\beta_{3\ell,r} \equiv \bar b_1 + \bar b_2\alpha + \bar b_3\alpha^2 = 0$, namely,

$$\begin{aligned}\alpha_{\beta z,3\ell,\pm} &= \frac{1}{2\bar b_3}\Big(-\bar b_2 \pm \sqrt{\Delta_2(\bar b_1,\bar b_2,\bar b_3)}\ \Big) \\ &= \frac{2\pi}{b_3}\Big(-b_2 \pm \sqrt{\Delta_2(b_1,b_2,b_3)}\ \Big) \ . \end{aligned} \tag{2.25}$$

where $\Delta_2(b_1,b_2,b_3) = b_2^2 - 4b_1b_3$. The analysis of the IR zero of $\beta_{3\ell}$ requires an consideration of the sign of $\Delta_2(b_1,b_2,b_3)$. The condition that $\beta_{3\ell}$ have an IR zero

requires, in particular, that its two zeros away from the origin be real, i.e., that $\Delta_2(b_1, b_2, b_3) \ge 0$. For a given G, R, and $N_f \in I$, so that b_1 and b_2 are fixed, this condition amounts to an upper bound on b_3, namely $b_3 \le b_2^2/(4b_1)$. Now, $b_2 \to 0$ at the lower end of the interval I, so that, insofar as one considers the analytic continuation of N_f from positive integers to positive real numbers, the above bound generically requires that $b_3 \le 0$ for $N_f \in I$. This is also required if $R = \Box$, and one studies the theory in the limit $N_c \to \infty$ and $N_f \to \infty$ with $r \equiv N_f/N_c$ fixed, since in this case there are discrete pairs of values (N_c, N_f) that enable one to approach arbitrarily close to the lower end of the interval I at $r = 34/13$ where $b_2 \to 0$. In order to preserve the existence of the two-loop IR zero at the three-loop level, one is thus motivated to restrict to schemes in which $b_3 \le 0$ for $N_f \in I$, and we will do so here. (The marginal case $b_3 = 0$ is not generic, since b_3 varies as a function of N_c and N_f, so we will not consider it further.)

Before proceeding, it is worthwhile to recall how the property $b_3 < 0$ for $N_f \in I$ arises in the $\overline{MS}$ scheme. In this scheme, b_3 is a quadratic function of N_f with positive coefficients of the N_f^2 term and the term independent of N_f. This coefficient b_3 vanishes, with sign reversal, at two values of N_f, denoted $N_{f,b3z,-}$ and $N_{f,b3z,+}$, given as Eq. (3.16) in [11], with $b_3 < 0$ for $N_{f,b3z,-} < N_f < N_{f,b3z,+}$ (and $b_3 > 0$ for $N_f < N_{f,b3z,-}$ and $N_f > N_{f,b3z,+}$. In [11] it was shown that in this scheme, for all of the representations considered there, namely, the fundamental ($\Box$), adjoint, and rank-2 symmetric ($\Box\Box$) and antisymmetric ($\begin{smallmatrix}\Box\\\Box\end{smallmatrix}$) tensor representations, $N_{f,b3z,-} < N_{f,b2z}$ and $N_{f,b3z,2} > N_{f,b1z}$, so that $b_3 < 0$ for all $N_f \in I$. For example, for fermions in the $R = \Box$ representation, (i) for $N_c = 2$, $N_{f,b3z,1} = 3.99 < N_{f,b2z} = 5.55$ and $N_{f,b3z,2} = 27.6 > N_{f,b1z} = 11$; (ii) for $N_c = 3$, $N_{f,b3z,1} = 5.84 < N_{f,b2z} = 8.05$, and $N_{f,b3z,2} = 40.6 > N_{f,b1z} = 16.5$; (iii) as $N_c \to \infty$, $N_{f,b3z,1} \to 1.911N_c$ while $N_{f,b2z} \to 2.615N_c$ and $N_{f,b3z,2} \to 13.348N_c$, while $N_{f,b1z} \to 5.5N_c$. In Table II we list the values of $\Delta_2(\bar{b}_1, \bar{b}_2, \bar{b}_3)$ with $\bar{b}_3$ calculated in the $\overline{MS}$ scheme, for the illustrative cases $N_c = 2,\ 3,\ 4$ and N_f in the respective I intervals. Since $b_3 < 0$ for $N_f \in I$ in this $\overline{MS}$ scheme, it follows that all of the entries in this table have $\Delta_2 > 0$.

Given that $b_3 < 0$ for $N_f \in I$, Eq. (2.25) can be rewritten as $\alpha = (2\pi/|b_3|)(-|b_2| \mp \sqrt{b_2^2 + 4b_1|b_3|})$. The solution with a $-$ sign in front of the square root is negative and hence unphysical; the other is positive and is $\alpha_{IR,3\ell}$, i.e.,

$$\alpha_{IR,3\ell} = \frac{2\pi}{|b_3|}\Big(-|b_2| + \sqrt{b_2^2 + 4b_1|b_3|}\ \Big) \ . \tag{2.26}$$

In [11] it was shown that in the $\overline{MS}$ scheme, for all $N_f \in I$, $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$. Here we demonstrate that this result holds more generally than just in the $\overline{MS}$ scheme. We prove that for arbitrary gauge group G, fermion representation R, and $N_f \in I$, in any scheme in which $b_3 < 0$ for $N_f \in I$ (which is thus guaranteed to preserve the IR zero present at the two-loop level), it follows that $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$. To prove this, we consider the difference

$$\alpha_{IR,2\ell} - \alpha_{IR,3\ell} = \frac{2\pi}{|b_2 b_3|}\Big[2b_1|b_3| + b_2^2 - |b_2|\sqrt{b_2^2 + 4b_1|b_3|}\ \Big] \ . \tag{2.27}$$

The expression in square brackets is positive if and only if

$$(2b_1|b_3| + b_2^2)^2 - b_2^2(b_2^2 + 4b_1|b_3|) > 0 \ . \tag{2.28}$$

This difference is equal to the nonnegative quantity $(2b_1b_3)^2$, which proves the inequality. Note that, since b_1 is nonzero for asymptotic freedom, this difference vanishes if and only if $b_3 = 0$, in which case $\alpha_{IR,3\ell} = \alpha_{IR,2\ell}$. We have therefore proved that

$$\alpha_{IR,3\ell} < \alpha_{IR,2\ell} \quad \text{if} \quad b_3 < 0 \quad \text{for} \quad N_f \in I. \tag{2.29}$$

As noted above, $\alpha_{IR,2\ell}$ is a monotonically decreasing function of $N_f \in I$. With $b_3 < 0$ for $N_f \in I$, this monotonicity property is also true of $\alpha_{IR,3\ell}$. As N_f increases from $N_{f,b2z}$ to $N_{f,b1z}$ in the interval I, $\alpha_{IR,3\ell}$ decreases from

$$\alpha_{IR,3\ell} = 4\pi\sqrt{\frac{b_1}{|b_3|}} \quad \text{at } N_f = N_{f,b2z} \tag{2.30}$$

to zero as $N_f \nearrow N_{f,b1z}$ at the upper end of this interval, vanishing like

$$\alpha_{IR,3\ell} = \frac{4\pi b_1}{|b_2|}\Big[1 - \frac{|b_3|b_1}{|b_2|^2} + O(b_1^2)\ \Big] \tag{2.31}$$

as $N_f \nearrow N_{f,b1z}$ and $b_1 \to 0$.

E. IR Zero of β at the Four-Loop Level

The four-loop β function is $\beta_{4\ell} = -2\alpha^2\beta_{4\ell,r}$, so $\beta_{4\ell}$ has three zeros away from the origin, at the roots of the cubic equation

$$\beta_{4\ell,r} \equiv \bar{b}_1 + \bar{b}_2\alpha + \bar{b}_3\alpha^2 + \bar{b}_4\alpha^3 = 0 \ , \tag{2.32}$$

(where $\beta_{n\ell,r}$ was given in Eq. (2.8)). These zeros were analyzed for the $\overline{MS}$ scheme in [11, 12]. Here we extend this analysis to a more general class of schemes that have $b_3 < 0$ for $N_f \in I$, and hence maintain at the three-loop level the IR zero of the scheme-independent two-loop β function.

The nature of the roots of Eq. (2.32) is determined by the sign of the discriminant $\Delta_3(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$, or equivalently,

$$\begin{aligned}\Delta_3 \equiv \Delta_3(b_1, b_2, b_3, b_4) \ &= \ b_2^2b_3^2 - 27b_1^2b_4^2 - 4(b_1b_3^3 + b_4b_2^3) \\ &+ \ 18b_1b_2b_3b_4 \ . \end{aligned} \tag{2.33}$$

The following properties of Δ_3 are relevant here: (i) if $\Delta_3 > 0$, then all of the roots of Eq. (2.32) are real; (ii) if $\Delta_3 < 0$, then Eq. (2.32) has one real root and a complex-conjugate pair of roots; (iii) if $\Delta_3 = 0$, then at least two of the roots of Eq. (2.32) coincide. Given the scheme-independent properties $b_1 > 0$ and $b_2 < 0$ (i.e., $N_f \in I$), and provided that $b_3 < 0$ for $N_f \in I$, we can write this discriminant as

$$\Delta_3(b_1, b_2, b_3, b_4) = b_2^2 b_3^2 - 27 b_1^2 b_4^2 + 4(b_1|b_3|^3 + b_4|b_2|^3) + 18 b_1|b_2||b_3|b_4 \ . \tag{2.34}$$

If b_4 were zero, then the zeros of $\beta_{4\ell}$ would coincide with those of $\beta_{3\ell}$, and the property that these are all real is in accord with the reduction

$$\Delta_3(b_1, b_2, b_3, 0) = b_3^2(b_2^2 + 4b_1|b_3|) = b_3^2 \Delta_2(b_1, b_2, b_3) \ , \tag{2.35}$$

which is positive.

Now consider nonzero b_4. First, assume that the scheme has the property that $b_4 > 0$. Then we can write Eq. (2.32) as

$$\bar{b}_1 - |\bar{b}_2|\alpha - |\bar{b}_3|\alpha^2 + \bar{b}_4\alpha^3 = 0 \quad \text{if} \quad b_4 > 0 \ . \tag{2.36}$$

From an application of the Descartes theorem on roots of algebraic equations, it follows that there are at most two (real) positive roots of this equation and at most one negative root. Moreover, from Eq. (2.12) we can deduce that in this case with $b_4 > 0$, in addition to the double zero at $\alpha = 0$, $\beta_{4\ell}$ has a zero at a negative value of α, so the upper bound on negative zeros from the Descartes theorem is saturated. Furthermore, since $\beta_{4\ell}$ is negative at large positive α, there are then two possibilities: either the two remaining zeros of Eq. (2.32) are a complex-conjugate pair, or else they are both real and positive.

If, on the other hand, the scheme is such that $b_4 < 0$, then we can write Eq. (2.32) as

$$\bar{b}_1 - |\bar{b}_2|\alpha - |\bar{b}_3|\alpha^2 - |\bar{b}_4|\alpha^3 = 0 \quad \text{if} \quad b_4 < 0 \ . \tag{2.37}$$

From a similar application of the Descartes theorem, we infer that there is at most one positive real root and at most two negative real roots of Eq. (2.37). From Eq. (2.11) we deduce that $\beta_{4\ell}$ has a zero at a positive real value of α. Depending on $|b_4|$, the other two roots of Eq. (2.37) may be real and negative or may form a complex-conjugate pair.

Combining the information from both the Descartes theorem and the discriminant Δ_3, we derive the following conclusions about the roots of Eq. (2.32) and hence the zeros of $\beta_{4\ell}$ aside from the double zero at $\alpha = 0$. As before, we assume that $b_2 < 0$ (i.e., $N_f \in I$) so that $\beta_{2\ell}$ has an IR zero, and also that the scheme is such that $b_3 < 0$ for $N_f \in I$, guaranteeing that this IR zero is maintained at the three-loop level. Then,

1. If $b_4 > 0$ and $\Delta_3 > 0$, then Eq. (2.32) has one negative and two positive real roots,
2. If $b_4 > 0$ and $\Delta_3 < 0$, then Eq. (2.32) has one negative root and a complex-conjugate pair of roots,
3. If $b_4 < 0$ and $\Delta_3 > 0$, then Eq. (2.32) has one positive root and two negative roots,
4. If $b_4 < 0$ and $\Delta_3 < 0$, then Eq. (2.32) has one positive root and a complex-conjugate pair of roots.

For a particular pair (N_c, N_f), the marginal case $\Delta_3 = 0$ might occur, and would mean that two of the roots of Eq. (2.32) are degenerate. Since this equation is a cubic, it would follow that all of the roots are real. If $\Delta_3 = 0$ and $b_4 > 0$, then Eq. (2.32) has one negative root and a positive root with multiplicity 2, while if $b_4 < 0$, then (2.32) has one positive root and a negative root with multiplicity 2.

It is reasonable to avoid schemes that lead to the outcome (2) above, with no real positive root of Eq. (2.32), since these fail to preserve the IR zero of the scheme-independent two-loop β function. Although the positivity of Δ_3 is not a necessary condition for this preserving of the IR zero, it is a sufficient condition. We thus investigate the conditions under which Δ_3 is positive. As shown via Eq. (2.35), if $b_4 = 0$, then $\Delta_3 > 0$. By continuity, for small $|b_4|$, Δ_3 remains positive, and there is only a small shift in the two zeros that were present in $\beta_{3\ell}$, together with the appearance of a new zero. Since the highest-degree term in $\Delta_3(b_1, b_2, b_3, b_4)$ involving $|b_4|$, namely $-27b_1^2 b_4^2$ is negative-definite, it follows that, other things being equal, for sufficiently large $|b_4|$, $\Delta_3(b_1, b_2, b_3, b_4)$ will decrease through zero and become negative. The two b_4 values at which $\Delta_3(b_1, b_2, b_3, b_4) = 0$ are

$$(b_4)_{\Delta 3z,\pm} = \frac{|b_2|(2b_2^2 + 9b_1|b_3|) \pm 2(b_2^2 + 3b_1|b_3|)^{3/2}}{27b_1^2} \tag{2.38}$$

Therefore, a sufficient condition for a scheme to be such that $\beta_{4\ell}$ preserves the IR zero that is present in $\beta_{2\ell}$ and $\beta_{3\ell}$ for $N_f \in I$ is

$$(b_4)_{\Delta 3z,-} < b_4 < (b_4)_{\Delta 3z,+} \ . \tag{2.39}$$

Note that at the lower end of the interval I, where $b_2 \to 0$, the interval (2.39) reduces to the upper bound $|b_4| < 2|b_3|^{3/2}/(27b_1)^{1/2}$.

For reference, in the $\overline{MS}$ scheme, b_4 is a cubic polynomial in N_f and is positive for $N_f \in I$ for $N_c = 2, 3$ but is negative in part of I for higher values of N_c (see Table I of [11], where N_c is denoted N). In Table II we list the values of $\Delta_3(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ with b_3 and b_4 calculated in the $\overline{MS}$ scheme, for the illustrative values $N_c = 2, 3, 4$ and values of N_f in the respective I intervals. For all of the four-loop entries in Table I, $\Delta_3(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4) > 0$, as is evident from the values listed explicitly in Table II, so these entries correspond to the case (1) in the list of possibilities for b_4 and Δ_3 given above.

Rather than calculating $\alpha_{IR,n\ell}$ directly from $\beta_{n\ell}$, a different approach is to use $\beta_{n\ell}$ to compute Padé approximants and then calculate zeros of these approximants.

As before, since one is interested in the zeros away from the origin, one extracts the factor $-2\alpha^2$ in Eq. (2.4) and analyzes the polynomial $\beta_{n\ell,r}$ in Eq. (2.8), of degree $n-1$ in α, depending on the n coefficients $\bar{b}_\ell$, $\ell = 1,..,n$. From this, one can construct a set of $[p,q]$ Padé approximants, i.e., rational functions, each with a numerator polynomial of degree p and a denominator polynomial of degree q in α, of the form $(\sum_{j=0}^p p_j z^j)/(\sum_{k=0}^q q_k z^k)$. Without loss of generality, one can divide numerator and denominator by q_0, so that, after redefinition of the coefficients, the $[p,q]$ Padé approximant to Eq. (2.8) is

$$[p,q] = \frac{\sum_{j=0}^p p_j \alpha^j}{1+\sum_{k=1}^q q_k \alpha^k} \ . \tag{2.40}$$

depending on the $p+q+1$ coefficients p_j with $j = 0,...,p$ and q_k with $k = 1,..,q$. These coefficients are determined by matching the Taylor series expansion of $[p,q]$ in α with the n coefficients $\bar{b}_\ell$, $\ell = 1,...,n$, so that $p+q = n-1$. Thus, from the four-loop beta function factor $\beta_{4\ell,r}$, one can construct two relevant approximants with $p+q = 3$, namely the [1,2] and [2,1] Padé approximants [27]. We did this in [11] and calculated the resultant unique IR zero from the [1,2] approximant and the relevant IR zero from [2,1], denoted $\alpha_{IR,4\ell,[1,2]}$ and $\alpha_{IR,4\ell,[2,1]}$, respectively. These were found to be close to the directly calculated IR zero, $\alpha_{IR,4\ell}$. From the known results for $\beta_{n\ell}$ with $n = 2,\ 3,\ 4$, one can make estimates of $\beta_{5\ell}$ by various methods, but since these only contain exact information up to the $n = 4$ loop level, we will not pursue this direction here.

As the $n = 4$ special case of the result discussed above, $\alpha_{IR,4\ell}$ decreases to zero as $N_f \nearrow N_{f,b1z}$. For the $R = \Box$ and for a given $N_f \in I$ where it is reliably calculable, $\alpha_{IR,4\ell}$ is slightly larger than $\alpha_{IR,3\ell}$, but the difference, $\alpha_{IR,4\ell} - \alpha_{IR,3\ell}$ is sufficiently small that $\alpha_{IR,4\ell}$ is smaller than $\alpha_{IR,2\ell}$. For higher fermion representations and N_f values where the IR zero is reliably calculable (i.e., not too close to the lower end of the interval I), the difference $\alpha_{IR,3\ell} - \alpha_{IR,4\ell}$ is again smaller in magnitude then the difference $\alpha_{IR,2\ell} - \alpha_{IR,3\ell}$ but may have either sign. Thus, where $\alpha_{IR,4\ell}$ is reliably calculable, it is smaller than $\alpha_{2,\ell}$. The finding that the fractional change in the location of the IR zero of β is reduced at higher-loop order agrees with the general expectation that calculating a quantity to higher order in perturbation theory should give a more stable and accurate result.

The scheme-dependence of $\alpha_{IR,n\ell}$ for $n \ge 3$ can be studied by carrying out scheme transformations, recalculating $\alpha'_{IR,n\ell}$ in the new scheme, and comparing with $\alpha_{IR,n\ell}$. This study was carried out in [14]. To be acceptable, a scheme transformation must satisfy a number of necessary conditions, such as mapping a a positive real α to a positive real α' and vice versa. Although these conditions can be satisfied easily in the vicinity of the ultraviolet fixed point of an asymptotically free theory at $\alpha = 0$, they are nontrivial and constitute significant restrictions on scheme transformations at an infrared fixed point [14]. For example, the scheme transformation $\alpha = \tanh\alpha'$, with inverse $\alpha' = (1/2)\ln[(1+\alpha)/(1-\alpha)]$, is acceptable for small α, in the vicinity of the UV fixed point of an asymptotically free gauge theory, but is not acceptable in the vicinity of an IR fixed point at $\alpha = \alpha_{IR} \sim O(1)$, since α can approach 1 from below, in which case α' diverges, and α can exceed 1, in which case α' is complex.

Scheme-dependence of higher-loop calculations is present not just in calculations of an IR zero of $\beta_{n\ell}$ at three- and higher-loop level, but also in higher-loop perturbative QCD calculations. The fact that the $\overline{MS}$ scheme is a reasonable one has been demonstrated, e.g., by the excellent fit that has been obtained to experimental data for $\alpha_s(\mu)$ with $\mu^2 = Q^2$ using this scheme [28]. There has been much work on optimized schemes for higher-order QCD calculations [29]. However, we note that two of the simplest scheme transformations that one might apply for QCD are not generally acceptable at an IR zero of β with $\alpha_{IR} \sim O(1)$. These are the scheme transformations denoted S_2 and S_3 in [14], which are constructed to render the leading scheme-dependent coefficient in the new scheme, b'_3, equal to zero. They are acceptable at the UV zero of β and hence in perturbative QCD applications, but are not, in general, acceptable in the vicinity of an IR zero with $\alpha_{IR} \sim O(1)$ because they can map a real positive α in the $\overline{MS}$ scheme to a negative or complex coupling in the transformed scheme, as was shown in [14].

F. Shift of IR Zero at $(n+1)$-loop Level

Here we derive a result on the direction of the shift in the IR zero of the β function when one increases the order of calculation of β from the n-loop level to the $(n+1)$-loop level, where $n \ge 2$. We assume, as before, that the theory is asymptotically free and that $b_2 < 0$ (i.e., $N_f \in I$), so that there is an IR zero of β at the two-loop level. We assume that the scheme-dependent coefficients b_ℓ with $\ell = 3,...,n+1$ are such that they preserve the existence of the IR zero of β at higher-loop level [30]. We focus here on values of α close to $\alpha_{IR,n\ell}$, where $d\beta_{n\ell}/d\alpha > 0$. Expanding $\beta_{n\ell}$ in a Taylor series expansion around $\alpha = \alpha_{IR,n\ell}$, with the abbreviation $\beta'_{IR,n\ell} \equiv d\beta_{n\ell}/d\alpha|_{\alpha_{IR,n\ell}}$ defined above, we write the general Eq. (2.17) explicitly in terms of n-loop quantities as

$$\beta_{n\ell} = \beta'_{IR,n\ell}\,(\alpha - \alpha_{IR,n\ell}) + O\Big((\alpha - \alpha_{IR,n\ell})^2\Big) \ . \tag{2.41}$$

Now let us calculate β to the next-higher-loop order, i.e., $\beta_{(n+1)\ell}$, and solve for the zero, $\alpha_{IR,(n+1)\ell}$, which corresponds to $\alpha_{IR,n\ell}$ (among the $n-1$ zeros of $\beta_{(n+1)\ell}$ away from the origin). To determine whether $\alpha_{IR,(n+1)\ell}$ is larger or smaller than $\alpha_{IR,n\ell}$, i.e., whether there is a shift to the right or left, consider the difference

$$\beta_{(n+1)\ell} - \beta_{n\ell} = -2\bar{b}_{n+1}\alpha^{n+2} \ . \tag{2.42}$$

In a scheme in which $b_{n+1} > 0$, this difference, evaluated at $\alpha = \alpha_{IR,n\ell}$, is negative, so, given that

$d\beta_{n\ell}/d\alpha|_{\alpha_{IR,n\ell}} > 0$, to compensate for this, the zero shifts to the right, whereas if $b_{n+1} < 0$, the difference is positive, so the zero shifts to the left. That is,

$$\text{If } b_{n+1} > 0 , \quad \text{then } \alpha_{IR,(n+1)\ell} > \alpha_{IR,n\ell}$$

$$\text{If } b_{n+1} < 0 , \quad \text{then } \alpha_{IR,(n+1)\ell} < \alpha_{IR,n\ell} . \quad (2.43)$$

(In a scheme with $b_{n+1} = 0$, obviously $\alpha_{IR,(n+1)\ell} = \alpha_{IR,n\ell}$). The application of this general result (2.43) is evident in the specific calculations in [11] at the three-loop and four-loop levels.

III. β FUNCTION STRUCTURE

At high scales in the UV, the β function is dominated by the leading quadratic term, $\beta \simeq -2\bar{b}_1\alpha^2 + O(\alpha^3)$. The calculation of the IR zero of $\beta_{n\ell}$ is important for investigating the UV to IR evolution of the theory. But, as discussed in the introduction, for a more detailed study of this evolution, one needs not just the value of the IR zero, $\alpha_{IR,n\ell}$, but the full curve of $\beta_{n\ell}$ for $\alpha \in I_\alpha$. Here we present calculations of three quantities that give further information about this curve, including (i) the value of α where $\beta_{n\ell}$ reaches its minimum for $\alpha \in I_\alpha$, $\alpha_{m,n\ell}$, (ii) the minimum value of $\beta_{n\ell}$ for $\alpha \in I_\alpha$, $(\beta_{n\ell})_{min}$; and the slope $\beta'_{IR,n\ell}$ at the IR zero of β, as defined in Eq. (2.16). The relevance of the third quantity to estimates of the dilaton mass in a quasiconformal gauge theory has been noted above. Our calculations are performed at the $n = 2$, $n = 3$, and $(n = 4)$-loop level.

A. Two-Loop Level

1. Position of Minimum in $\beta_{2\ell}$ for $\alpha \in I_\alpha$

At the two-loop level, given that $b_2 < 0$ so that $\beta_{2\ell}$ function has an IR zero, the derivative $d\beta_{2\ell}/d\alpha = -2\alpha(2\bar{b}_1 + 3\bar{b}_2\alpha)$ vanishes at $\alpha = \alpha_{m,2\ell}$, where

$$\alpha_{m,2\ell} = -\frac{2\bar{b}_1}{3\bar{b}_2} = -\frac{8\pi b_1}{3b_2} = \frac{8\pi b_1}{3|b_2|} . \qquad (3.1)$$

Explicitly,

$$\alpha_{m,2\ell} = \frac{8\pi(11C_A - 4T_fN_f)}{3[4(5C_A + 3C_f)T_fN_f - 34C_A^2]} . \qquad (3.2)$$

2. Minimum Value of $\beta_{2\ell}$ for $\alpha \in I_\alpha$

At $\alpha = \alpha_{m,2\ell}$, $\beta_{2\ell}$ reaches its minimum physical value for $\alpha \in I_\alpha$, namely

$$(\beta_{2\ell})_{min} = -\frac{8\bar{b}_1^3}{27\bar{b}_2^2} = -\frac{32\pi b_1^3}{27b_2^2}$$

$$= -\frac{32\pi(11C_A - 4T_fN_f)^3}{81[34C_A^2 - 4(5C_A + 3C_f)T_fN_f]^2} . \qquad (3.3)$$

Note that

$$\alpha_{m,2\ell} = \frac{2}{3}\alpha_{IR,2\ell} . \qquad (3.4)$$

3. Slope of $\beta_{2\ell}$ at $\alpha_{IR,2\ell}$

The derivative $d\beta_{2\ell}/d\alpha$ evaluated at $\alpha = \alpha_{IR,2\ell}$ is

$$\beta'_{IR,2\ell} = -\frac{2\bar{b}_1^2}{\bar{b}_2} = -\frac{2b_1^2}{b_2} = \frac{2b_1^2}{|b_2|}$$

$$= \frac{2(11C_A - 4T_fN_f)^2}{3[4(5C_A + 3C_f)T_fN_f - 34C_A^2]} , \qquad (3.5)$$

which is positive for $N_f \in I$.

As descriptors of the shape and structure of the β function, the quantities $\alpha_{m,n\ell}$, $(\beta_{n\ell})_{min}$, and $\beta'_{IR,n\ell}$ are interrelated. Thus, if one makes a rough, linear (*lin.*) approximation to the β function in the interval from $\alpha = \alpha_{m,n\ell}$ to $\alpha = \alpha_{IR,n\ell}$, then this slope would be

$$\frac{\Delta\beta_{n\ell;lin.}}{\Delta\alpha} = \frac{-(\beta_{n\ell})_{min}}{\alpha_{IR,n\ell} - \alpha_{m,n\ell}} . \qquad (3.6)$$

For example, in the $(n = 2)$-loop case, substituting the values of $(\beta_{2\ell})_{min}$, $\alpha_{IR,2\ell}$, and $\alpha_{m,2\ell}$, this approximation yields

$$\frac{\Delta\beta_{2\ell;lin.}}{\Delta\alpha} = -\frac{8\bar{b}_1^2}{9\bar{b}_2} = \frac{8b_1^2}{9|b_2|} , \qquad (3.7)$$

which exhibits the same dependence on the input coefficients b_1 and b_2, with a somewhat smaller coefficient, 8/9 rather than the coefficient 2 in the exact two-loop expression, $\beta'_{IR,2\ell}$, in Eq. (3.5).

In Tables I, III, and IV we list numerical values of $\alpha_{m,n\ell}$, $(\beta_{n\ell})_{min}$, and $\beta'_{IR,n\ell}$ for fermions in the $R = \Box$ representation of SU(N_c), for some illustrative cases of N_c and, for each N_c, values of N_f in the respective intervals I. As illustrations, we show plots of $\beta_{n\ell}$ in Fig. 1 for $N_c = 2$ and $N_f = 8$ and in Fig. 2 for $N_c = 3$ and $N_f = 12$ as functions of α. The results in the tables and figures are given for the quantities evaluated at the $n = 2$, $n = 3$, and $n = 4$ loop levels. The $n = 3$ and $n = 4$ loop results will be discussed further below.

B. Three-Loop Level

1. Position of Minimum in $\beta_{3\ell}$ for $\alpha \in I_\alpha$

Here we assume a scheme in which $b_3 \neq 0$, since if one is working with a scheme in which $b_3 = 0$, then $\beta_{3\ell} = \beta_{2\ell}$, so the analysis of the three-loop β function reduces to that of the two-loop β function discussed above. Furthermore,

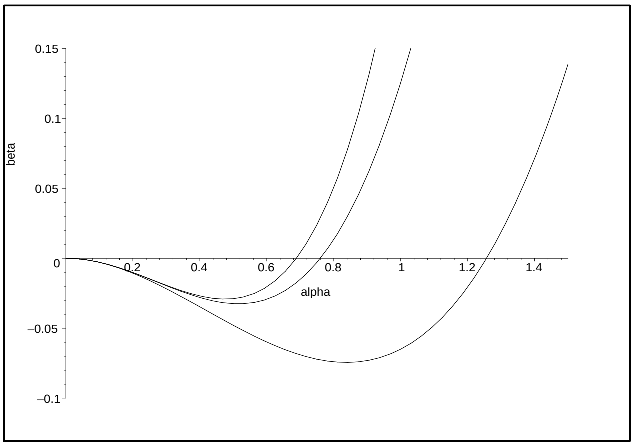

FIG. 1: Plot of the n-loop β function $\beta_{n\ell}$ as a function of α for $n = 2,\ 3,\ 4$ and $N_c = 2$, $N_f = 8$ with fermions in the fundamental representation. At a given value of α, the curves, from bottom to top, are for $\beta_{2\ell}$, $\beta_{4\ell}$, and $\beta_{3\ell}$, respectively. See text for further details.

for the reasons explained above, we restrict to schemes in which $b_3 < 0$ for $N_f \in I$. The derivative $d\beta_{3\ell}/d\alpha = -2\alpha(2\bar{b}_1 + 3\bar{b}_2\alpha + 4\bar{b}_3\alpha^2)$ is zero at $\alpha = 0$ and at the two other points,

$$\alpha = \frac{1}{8\bar{b}_3}\Big[-3\bar{b}_2 \pm \sqrt{9\bar{b}_2^2 - 32\bar{b}_1\bar{b}_3}\ \Big] \ . \qquad (3.8)$$

This can be written as $\alpha = \pi(2|b_3|)^{-1}(-3|b_2| \mp \sqrt{9b_2^2 + 32b_1|b_3|})$. The critical point corresponding to the $-$ sign in front of the square root is negative and hence unphysical, while the critical point corresponding to the $+$ sign in front of the square root is $\alpha_{m,3\ell}$, i.e.,

$$\alpha_{m,3\ell} = \frac{\pi}{2|b_3|}\Big[-3|b_2| + \sqrt{9b_2^2 + 32b_1|b_3|}\ \Big] \ . \qquad (3.9)$$

A general inequality is

$$\alpha_{m,3\ell} < \alpha_{m,2\ell} \ . \qquad (3.10)$$

We prove this by examining the difference

$$\alpha_{m,2\ell} - \alpha_{m,3\ell} = \frac{\pi}{6|b_2 b_3|}\Big[16b_1|b_3| + 9b_2^2 - 3|b_2|\sqrt{9b_2^2 + 32b_1|b_3|}\ \Big] \qquad (3.11)$$

The condition that this is positive is equivalent to the condition that the square of the polynomial term in the numerator of Eq. (3.11) minus the square of the term in this numerator involving the square root is positive. This difference of squares is equal to $256b_1^2b_3^2$, which is positive. This proves the inequality (3.10).

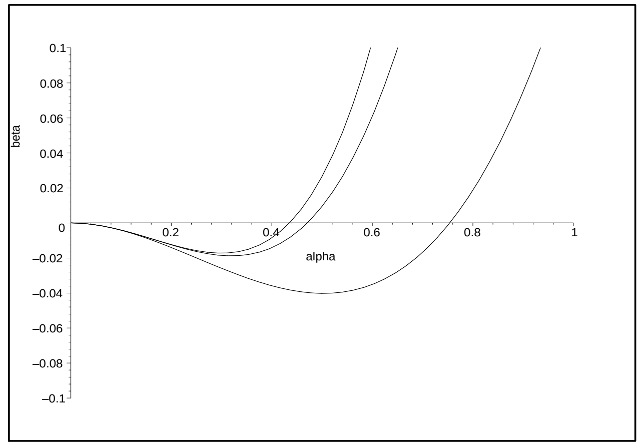

FIG. 2: Plot of the n-loop β function $\beta_{n\ell}$ as a function of α for $n = 2,\ 3,\ 4$ and the illustrative case $N_c = 3$, $N_f = 12$ with fermions in the fundamental representation. At a given value of α, the curves, from bottom to top, are for $\beta_{2\ell}$, $\beta_{4\ell}$, and $\beta_{3\ell}$, respectively. See text for further details.

2. *Minimum Value of $\beta_{3\ell}$ for $\alpha \in I_\alpha$*

At $\alpha = \alpha_{\beta_{min},3\ell}$, $\beta_{3\ell}$ reaches its minimum value for $\alpha \in I_\alpha$, namely

$$(\beta_{3\ell})_{min} = \frac{\pi}{64|b_3|^3}\Big[-\Big(144b_1b_2^2|b_3| + 128b_1^2b_3^2 + 27b_2^4\Big) + |b_2|\Big(9b_2^2 + 32b_1|b_3|\Big)^{3/2}\ \Big] \ . \qquad (3.12)$$

Note that one can write $(\beta_{3\ell})_{min}$ in terms of the $\bar{b}_\ell$ coefficients by replacing each b_ℓ in Eq. (3.16) by the corresponding $\bar{b}_\ell$ and dividing the overall expression by 4π. Since $(\beta_{n\ell})_{min} < 0$, it is convenient to deal with the magnitudes $|(\beta_{n\ell})_{min}|$. We find the following general inequality: for a given G, R, and $N_f \in I$, in a scheme that has $b_3 < 0$ and hence maintains the existence of the IR zero in $\beta_{2\ell}$,

$$|(\beta_{3\ell})_{min}| < |(\beta_{2\ell})_{min}| \ . \qquad (3.13)$$

To prove this, we consider the difference

$$|(\beta_{2\ell})_{min}| - |(\beta_{3\ell})_{min}| = \frac{\pi}{1728 b_2^2 |b_3|^3} \Big[2048 b_1^3 |b_3|^3 + 27 b_2^2 (144 b_1 b_2^2 |b_3| + 128 b_1^2 b_3^2 + 27 b_2^4) - 27 |b_2|^3 (9 b_2^2 + 32 b_1 |b_3|)^{3/2} \Big] . \tag{3.14}$$

The positivity of this difference is equivalent to the positivity of the square of the polynomial terms in the numerator minus the square of the term in the numerator involving the radical. This difference of squares is equal to

$$8192 b_1^3 |b_3|^3 \Big(512 b_1^3 |b_3|^3 + 3402 b_2^4 b_1 |b_3| + 1728 b_2^2 b_1^2 b_3^2 + 729 b_2^6 \Big) . \tag{3.15}$$

This expression is manifestly positive-definite, which proves the inequality (3.13).

3. Slope of $\beta_{3\ell}$ at $\alpha_{IR,3\ell}$

The derivative of $\beta_{3\ell}$ at $\alpha = \alpha_{IR,3\ell}$ is

$$\beta'_{IR,3\ell} \;=\; = \frac{1}{|b_3|^2} \Big[-4|b_2|(b_2^2 + b_1|b_3|)$$

$$+ \;\; (b_2^2 + 2b_1|b_3|)\sqrt{b_2^2 + 4b_1|b_3|} \;\; \Big] . \tag{3.16}$$

That this is positive follows from the fact that the square of the term involving the square root minus the square of $-(4b_1|b_2||b_3| + |b_2|^3)$ in the brackets is the manifestly positive quantity $4b_1^2 b_3^2(b_2^2 + 4b_1|b_3|)$. Owing to the homogeneity properties, to express $\beta'_{IR,3\ell}$ in terms of the $\bar{b}_\ell$ coefficients, one simply replaces each b_ℓ in Eq. (3.16) by the corresponding $\bar{b}_\ell$.

A general inequality is that for a given G, R, and $N_f \in I$, in a scheme with $b_3 < 0$, which is thus guaranteed to maintain the existence of the IR zero in $\beta_{2\ell}$ at the three-loop level,

$$\beta'_{IR,3\ell} < \beta'_{IR,2\ell} . \tag{3.17}$$

To prove this, we examine the difference

$$\beta'_{IR,2\ell} - \beta'_{IR,3\ell} = \frac{1}{|b_2| b_3^2} \Big[2b_1^2 b_3^2 + 4|b_2|(b_2^2 + b_1|b_3|) - |b_2|(b_2^2 + 2b_1|b_3|)\sqrt{b_2^2 + 4b_1|b_3|} \;\Big] . \tag{3.18}$$

The positivity of this difference is equivalent to the positivity of the square of the polynomial terms in the numerator minus the square of the term in the numerator involving the square root, which is

$$4b_1^4 b_3^4 + 12 b_2^4 b_1^2 b_3^2 + 15 b_2^8 + 24 b_2^6 b1 |b_3| . \tag{3.19}$$

This is manifestly positive-definite, which proves the inequality (3.17).

The shifts in the values of the IR zero, $\alpha_{IR,n\ell}$, the position of the minimum in $\beta_{n\ell}$, the value of $\beta_{n\ell}$ at the minimum, and the slope of $\beta_{n\ell}$ at $\alpha = \alpha_{n,\ell}$ are evident from Tables I, III, and IV and Figs. 1 and 2.

C. Four-Loop Level

The derivative $d\beta_{3\ell}/d\alpha = -2\alpha(2\bar{b}_1 + 3\bar{b}_2\alpha + 4\bar{b}_3\alpha^2 + 5\bar{b}_4\alpha^3)$ is zero at $\alpha = 0$ and at the three other points given by the zeros of the cubic equation $2\bar{b}_1 + 3\bar{b}_2\alpha + 4\bar{b}_3\alpha^2 + 5\bar{b}_4\alpha^3 = 0$. We have calculated these critical points, evaluated $\beta_{4\ell}$ at its minimum physical value, and also evaluated the derivative $d\beta_{4\ell}/d\alpha$ at $\alpha = \alpha_{IR,4\ell}$. We give the numerical results for $\alpha_{m,4\ell}$, $(\beta_{4\ell})_{min}$, and $\beta'_{IR,4\ell}$ in Tables I, III, and IV. These four-loop structural results are also evident in Figs. 1 and 2.

In addition to the results that we have proved above, we note some others here. As stated above, numerical results for three- and four-loop structural quantities were calculated in the $\overline{MS}$ scheme. First, although the ratios $\alpha_{m,3\ell}/\alpha_{IR,3\ell}$ and $\alpha_{m,4\ell}/\alpha_{IR,4\ell}$ are not constants as functions of N_f, they do not differ very much from the two-loop ratio, which is a constant, namely, 2/3, as given in Eq. (3.4). With fermions in the $\square$ representation, for a given N_c and $N_f \in I$, $\alpha_{m,4\ell}$ is slightly larger than $\alpha_{m,3\ell}$, but still substantially smaller than $\alpha_{m,2\ell}$, just as is true of the corresonding $\alpha_{IR,n\ell}$ quantities. Moreover, for a given G and loop order n, $\alpha_{m,n\ell}$ is a monotonically decreasing function of $N_f \in I$ and vanishes as $N_f \nearrow N_{f,b1z}$ and $b_1 \to 0$.

IV. SOME PROPERTIES OF γ_m

The anomalous dimension γ_m for the fermion bilinear $\bar{\psi}\psi$ describes the scaling properties of this operator and

can be expressed as a series in a or equivalently, α:

$$\gamma_m = \sum_{\ell=1}^{\infty} c_\ell \, a^\ell = \sum_{\ell=1}^{\infty} \bar{c}_\ell \, \alpha^\ell \ , \tag{4.1}$$

where $\bar{c}_\ell = c_\ell/(4\pi)^\ell$ is the ℓ-loop series coefficient. The coefficient c_1 is scheme-independent, while c_ℓ for $\ell \ge 2$ are scheme-dependent. The c_ℓ coefficients have been calculated up to four-loop order in the $\overline{MS}$ scheme [7]. We list c_ℓ for $\ell = 1, \ 2, \ 3$ in Appendix A. We denote the n-loop expression for γ_m as a series in α, evaluated at the n-loop IR zero of β, $\alpha = \alpha_{IR,n\ell}$, as $\gamma_{IR,n\ell}$.

In [11], we calculated $\gamma_{IR,n\ell}$ up to $(n = 4)$-loop order in the $\overline{MS}$ scheme. An important result was that we found a substantial reduction in γ_{IR} going from the two-loop to three-loop level for all of the fermion representations that were considered. The difference going from three- to four-loop level, $\gamma_{IR,3\ell} - \gamma_{IR,4\ell}$, was found to be smaller and could be of either sign, depending on the representation and value of N_f. The resultant $\gamma_{IR,4\ell}$ was thus substantially smaller than $\gamma_{IR,2\ell}$.

One may investigate the reduction $\gamma_{IR,3\ell} < \gamma_{IR,2\ell}$ found in [11] further. To do this for a given gauge group G and fermion representation R, we assume that $N_f \in I$, so that the theory has an IR zero of $\beta_{2\ell}$, and, further, that the scheme is such that $b_3 < 0$ for $N_f \in I$, so that this IR zero is guaranteed to be maintained at the three-loop level. We will use the resultant property that $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$. Let us consider the difference $\gamma_{IR,2\ell} - \gamma_{IR,3\ell}$. This is given by

$$\gamma_{IR,2\ell} - \gamma_{IR,3\ell} = \bar{c}_1(\alpha_{IR,2\ell} - \alpha_{IR,3\ell}) + \bar{c}_2(\alpha_{IR,2\ell}^2 - \alpha_{IR,3\ell}^2) - \bar{c}_3\alpha_{IR,3\ell}^3 \ . \tag{4.2}$$

The (scheme-independent) coefficient c_1 is positive, so that, since $\alpha_{IR,2\ell} - \alpha_{IR,3\ell} > 0$, it follows that the first term on the right-hand side of Eq. (4.2) is positive. The factor $(\alpha_{IR,2\ell}^2 - \alpha_{IR,3\ell}^2)$ in the second term is also positive. The coefficient c_2 is scheme-dependent, so the analysis of this term necessarily involves a choice of scheme, as does the analysis of the third term. We next prove that in the $\overline{MS}$ scheme, $c_2 > 0$ for all of the representations considered in [11], so that this second term is positive. To show this, we begin with the $\square$ representation, for which

$$(c_2)_{fund,\overline{MS}} = \frac{(N_c^2-1)(203N_c^2 - 9 - 20N_cN_f)}{192N_c^2} \ . \tag{4.3}$$

The first factor in the numerator, $N_c^2 - 1$, is obviously positive for all physical N_c. The second factor is positive for $N_f < N_{f,c2z}$, where

$$N_{f,c2z} = \frac{203N_c^2 - 9}{20N_c} \ . \tag{4.4}$$

This is larger than the upper bound on N_f from asymptotic freedom, $N_{f,b1z}$, as is clear from the difference

$$N_{f,c2z} - N_{f,b1z} = \frac{3(31N_c^2 - 3)}{20N_c} > 0 \ , \tag{4.5}$$

so that $c_{2,fund,\overline{MS}} > 0$ for $N_f \in I$ (actually for all physical N_f). This provides an analytic understanding of the numerical results in Table V of [11], which indicated that $c_2 > 0$ for all N_c and N_f considered there.

We next consider the case of fermions in the adjoint representation, for which

$$c_{2,adj,\overline{MS}} = \frac{N_c^2(53 - 10N_f)}{24} \ . \tag{4.6}$$

This is positive for $N_f < 53/10$, which is larger than the upper bound on N_f for this representation from the requirement of asymptotic freedom, namely $N_{f,b1z} = 11/4 = 2.75$, so again, $c_2 > 0$ for all N_f for the adjoint representation in the $\overline{MS}$ scheme.

Finally, we consider the case of fermions in the symmetric or antisymmetric rank-2 tensor representation, denoted $S2$ and $A2$, respectively, with Young tableaux $\square\square$ and $\begin{smallmatrix}\square\\\square\end{smallmatrix}$. Owing to the fact that the $A2$ representation of SU(2) is the singlet, it is understood that $N_c \ge 3$ in this case. Since various formulas are similar for these two representations, with appropriate reversals of signs of certain terms, it is convenient to give them in a unified manner, with $T2$ referring to $S2$ and $A2$ together. We have

$$c_{2,T2,\overline{MS}} = \frac{(N_c \pm 2)(N_c \mp 1)\Big[109N_c^2 \pm 9N_c - 18 - 10N_c(N_c \pm 2)N_f\Big]}{48N_c^2} \ , \tag{4.7}$$

where the upper (lower) sign applies for the $S2$ ($A2$) representation, respectively. In the numerator of this expression, the factor $(N_c \pm 2)(N_c \mp 1)$ is obviously positive for the relevant values of N_c, so one next examines the factor $[109N_c^2 \pm 9Nc - 18 - 10N_c(Nc \pm 2)N_f]$. This is positive for $N_f < N_{f,c2T2z}$, where

$$N_{f,c2T2z} = \frac{106N_c^2 \pm 9N_c - 18}{10N_c(N_c \pm 2)} \ . \tag{4.8}$$

As for the other representations, $N_{f,c2T2z}$ is larger than the respective upper bound on N_f from the requirement of asymptotic freedom, $N_{f,b1z,T2}$,

$$N_{f,b1z,T2} = \frac{11(N_c}{2(N_c \pm 2)} \ . \tag{4.9}$$

This is proved by considering the difference

$$N_{f,c2T2z} - N_{f,b1z,T2} = \frac{3(17N_c^2 \pm 3N_c - 6)}{10N_c(N_c \pm 2)} \tag{4.10}$$

This difference is positive for

$$N_c \ge \frac{\mp 3 + \sqrt{417}}{34} \ , \tag{4.11}$$

i.e., 0.5124 for $S2$ and 0.6888 for $A2$, and hence for all physical N_c. Therefore, this proves that $c_2 > 0$ for all relevant $N_f < N_{f,b1z}$ and, in particular, for all N_f in the respective intervals I for these theories with fermions in the symmetric or antisymmetric rank-2 representation.

We have thus proved that for the $\overline{MS}$ scheme, for all of the representations considered in [11], the first two terms in the difference $\gamma_{IR,2\ell} - \gamma_{IR,3\ell}$ are both positive. We have also investigated the contribution of the third term. By analytic methods similar to those exhibited above, we find that this third term also makes a positive contribution to the difference in Eq. (4.2), i.e., $c_3 < 0$ for $N_f \in I$, in most, although not all, cases. For example, for $G = \mathrm{SU}(N_c)$ and fermions in the $\Box$ representation, $c_3 < 0$ for all N_c up to $N_c = 15$ and integer N_f values in the respective intervals I. This includes all of the cases of N_c and $N_f \in I$ for which numerical results were given in [11] and thus gives an analytic understanding of those results. For $N_c = 16$, the interval I is $42 \le N_f \le 87$, and $c_3 < 0$ for all of these values of N_f except the lowest one, $N_f = 42$, where $c_3 > 0$. Similar comments apply for larger N_c.

V. SUPERSYMMETRIC GAUGE THEORY

A. IR Zeros of β

It is of interest to give some corresponding results on properties of the β function and associated UV to IR evolution in an asymptotically free, $\mathcal{N} = 1$ supersymmetric gauge theory with vectorial chiral superfield content Φ_i, $\tilde\Phi_i$, $i = 1, ..., N_f$ in the R, $\bar R$ representations, respectively. A number of exact results have been derived describing the infrared properties of the theory in [31, 32]. Thus, one can compare findings from perturbative calculations with these exact results, and this was done in [13]. The β function of the theory has the form

$$\beta_s = \frac{d\alpha}{dt} = -2\alpha \sum_{\ell=1}^{\infty} b_{\ell,s} a^{\ell} = -2\alpha \sum_{\ell=1}^{\infty} \bar b_{\ell,s} \alpha^{\ell} \ , \tag{5.1}$$

where we use the subscript s, standing for supersymmetric, to avoid confusion with the corresponding quantities in the nonsupersymmetric theory, and $\bar b_{\ell,s} = b_{\ell,s}/(4\pi)^{\ell}$. The beta function calculated to n-loop order is denoted $\beta_{n\ell,s}$. For values of N_f for which $\beta_{n\ell,s}$ has an IR zero, we denote this as $\alpha_{IR,n\ell,s}$. In addition to the scheme-independent coefficients $b_{1,s}$ and $b_{2,s}$, calculated in [33] and [34], respectively, the three-loop coefficient, $b_{3,\ell}$, has been calculated in [35] in the dimensional reduction ($\overline{DR}$) scheme [36]. Calculations of $\alpha_{IR,n\ell,s}$ and corresponding values of the anomalous dimension for the bilinear chiral superfield operator $\Phi\tilde\Phi$ were given in [13] up to the maximal order to which b_ℓ and the coefficients of the anomalous dimension had been calculated, namely the three-loop level.

We recall that, since $b_1 = 3C_A - 2T_f N_f$ [33], the upper bound on N_f for the theory to be asymptotically free is

$$N_f < N_{f,b1z,s} \ , \tag{5.2}$$

where

$$N_{f,b1z,s} = \frac{3C_A}{2T_f} \ . \tag{5.3}$$

The two-loop β function coefficient is $b_{2,s} = 6C_A^2 - 4(C_A + 2C_f)T_f N_f$ [34], which decreases through positive values and passes through zero, reversing sign, as N_f increases through

$$N_{f,b2z,s} = \frac{3C_A^2}{2T_f(C_A + 2C_f)} \ . \tag{5.4}$$

Since $N_{f,b2z,s} < N_{f,b1z,s}$, there is always an interval of values of N_f, namely

$$I_s : \quad N_{f,b2z,s} < N_f < N_{f,b1z,s} \ , \tag{5.5}$$

in which the two-loop β function for this theory has an IR zero for N_f. The value of this two-loop IR zero is

$$\begin{aligned} \alpha_{IR,2\ell,s} &= -\frac{4\pi b_{1,s}}{b_{2,s}} \\ &= \frac{2\pi(3C_A - 2T_f N_f)}{2(C_A + 2C_f)T_f N_f - 3C_A^2} \ . \end{aligned} \tag{5.6}$$

In particular, for chiral superfields in the $\Box$ and $\bar\Box$ representations, $N_{f,b1z,s} = 3N_c$ and $N_{f,b2z,s} = 3N_c^3/[2N_c^2 - 1]$, so

$$I_s : \frac{3N_c^3}{2N_c^2 - 1} < N_f < \frac{3N_c}{2} \quad \text{for } R = \Box \ . \tag{5.7}$$

In this case, the exact value of N_f at the lower end of the IR-conformal, non-Abelian Coulomb phase was determined in Ref. [32] to be

$$N_{f,cr,s} = \frac{3N_c}{2} \quad \text{for } R = \Box \ . \tag{5.8}$$

Here, since $N_{f,b2z,s} > N_{f,cr,s}$, the coefficient $b_{2,s}$ passes through zero and reverses sign in the interior of the non-Abelian Coulomb phase. Consequently, as was noted in [13], for this case of chiral superfields in the $\Box$ and $\bar{\Box}$ representations, one cannot study the IR zero of $\beta_{2\ell}$ throughout the entirety of this phase. The generalization of $N_{f,cr,s}$ to higher representations has been given as [37]

$$N_{f,cr,s} = \frac{3C_A}{2T_f} \ . \tag{5.9}$$

Assuming that γ_m saturates its upper bound of 1 in this supersymmetric theory as $N_f \searrow N_{f,cr,s}$, Eq. (5.8) agrees with the result obtained via a closed-form solution for β [31] (see also [38]). Note that the fact that the scheme used in [31] is different from the $\overline{DR}$ scheme does not affect this, since $N_{f,cr,s}$ is a physical quantity.

At the three-loop level, $\beta_{3\ell,s} = -2\alpha^2\beta_{3\ell,r,s}$, where $\beta_{n\ell,r}$ is given by Eq. (2.8) with $\bar{b}_\ell$ replaced by $\bar{b}_{\ell,s}$. One makes use of the result [35]

$$\begin{aligned} b_{3,s} \ &= \ 21C_A^3 + 4(-5C_A^2 - 13C_AC_f + 4C_f^2)T_fN_f \\ &+ \ 4(C_A + 6C_f)T_f^2N_f^2 \end{aligned} \tag{5.10}$$

in the $\overline{DR}$ scheme. The three-loop IR zero of β_s, $\alpha_{IR,3\ell,s}$, was calculated and compared with $\alpha_{IR,2\ell,s}$ in [13]. One can prove various inequalities similar to those that we have proved above for a non-supersymmetric gauge theory. We illustrate one of these, concerning the relative size of $\alpha_{IR,2\ell,s}$ and $\alpha_{IR,3\ell,s}$ for chiral superfields in the $\Box$ and $\bar{\Box}$ representations. We begin by noting that $b_{3,s}$ in the $\overline{DR}$ scheme is a quadratic function of N_f which is positive for small N_f, and, as N_f increases, passes through zero, becoming negative, at a value denoted $N_{f,b3z,-,s}$, reaches a minimum, and then passes through zero again at $N_f = N_{f,b3z,+,s}$, and is positive for larger N_f. In general,

$$N_{f,b3z,\pm,s} = \frac{5C_A^2 + 13C_AC_f - 4C_f^2 \pm \sqrt{R_s}}{2T_f(C_A + 6C_f)} \ , \tag{5.11}$$

where

$$R_s = 4C_A^4 + 4C_A^3C_f + 129C_A^2C_f^2 - 104C_AC_f^3 + 16C_f^4 \ . \tag{5.12}$$

For the $\Box$ or $\bar{\Box}$ representation, this reduces to

$$N_{f,b3z,\pm,s} = \frac{21N_c^4 - 9N_c^2 - 2 \pm \sqrt{R_{s,fund}}}{2N_c(4N_c^2 - 3)} \ , \tag{5.13}$$

where

$$R_{s,fund} = 105N_c^8 - 126N_c^6 - 3N_c^4 + 36N_c^2 + 4 \ . \tag{5.14}$$

(Note that $R_{s,fund}$ is positive-definite, vanishing at eight complex values of N_f.) To prove that $\alpha_{IR,2\ell,s} < \alpha_{IR,3\ell,s}$ for this case, it suffices to show that $b_{3,s} < 0$ for $N_f \in I_s$, since then one can apply the same proof that we used for Eq. (2.29). To show that $b_{3,s} < 0$ for $N_f \in I_s$, we will demonstrate that $N_{f,b3z,-,s} < N_{f,b2z,s}$ and $N_{f,b3z,+,s} > N_{f,b1z,s}$. First, for this case of R equal to the $\Box$ representation, we consider the difference

$$N_{f,b2z,s} - N_{f,b3z,-,s} = \frac{-18N_c^6 + 21N_c^4 - 5N_c^2 - 1 + (2N_c^2 - 1)\sqrt{R_{s,fund}}}{2N_c(2N_c^2 - 1)(4N_c^2 - 3)} \ . \tag{5.15}$$

Although the polynomial term in the numerator of (5.15) is negative, it is smaller than the term involving the square root. To show this, we observe that the square of the term involving the square root minus the square of the polynomial term in the numerator is $24Nc^4(N_c^2 + 1)(N_c^2 - 1)^2(4N_c^2 - 3)$. This is positive for all physical N_c, proving that $N_{f,b2z,s} > N_{f,b3z,-,s}$ for this case. Next, we consider the difference

$$N_{f,b3z,+,s} - N_{f,b1z} = \frac{-3N_c^4 + 9N_c^2 - 2 + \sqrt{R_{s,fund}}}{2N_c(4N_c^2 - 3)} \ . \tag{5.16}$$

Although the polynomial term in the numerator is negative for physical N_c, it is smaller than the square root, as is shown by the fact that the difference of the square of the square root term minus the square of the polynomial term is $24N_c^2(N_c^4 - 1)(4N_c^2 - 3)$, which is positive. So we have proved that for this case, $N_{f,b3z,-,s} < N_{f,b2z,s}$ and $N_{f,b3z,+,s} > N_{f,b1z,s}$. In turn, this proves that for this case with N_f chiral superfields in the $\Box$ and $\bar{\Box}$ representations, in the $\overline{DR}$ scheme, $b_{3,s} < 0$ for $N_f \in I_s$, and hence

$$\alpha_{IR,3\ell,s} < \alpha_{IR,2\ell,s} \ . \tag{5.17}$$

This inequality follows by the same type of proof as the one that we gave for Eq. (2.29).

B. Structural Properties of β_s

Because one has exact, nonperturbative results available for this theory, we will be brief in our discussion of structural properties of the β function. The value of α where $\beta_{2\ell,s}$ has zero slope and a minimum in the interval

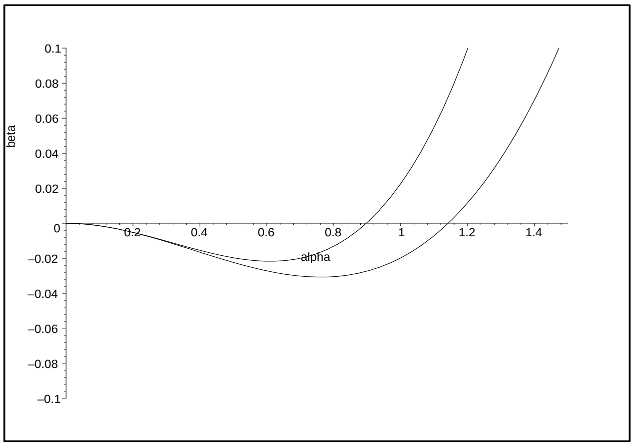

FIG. 3: Plot of the n-loop β function $\beta_{n\ell,s}$ for an SU(N_c) gauge theory with $\mathcal{N}=1$ supersymmetry, as a function of α, for $n=2$ and $n=3$ loops and $N_c=2$, $N_f=5$ with chiral superfields in the fundamental representation. The lower and upper curves correspond to $\beta_{2\ell,s}$ and $\beta_{3\ell,s}$, respectively. See text for further details.

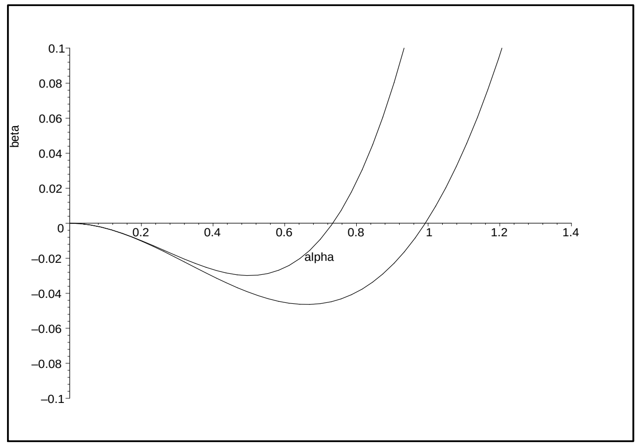

FIG. 4: Plot of the n-loop β function $\beta_{n\ell,s}$ for an SU(N_c) gauge theory with $\mathcal{N}=1$ supersymmetry, as a function of α, for $n=2$ and $n=3$ loops and $N_c=3$, $N_f=7$ with chiral superfields in the fundamental representation. The lower and upper curves correspond to $\beta_{2\ell,s}$ and $\beta_{3\ell,s}$, respectively. See text for further details.

(1.1) is given by Eq. (3.1) as

$$\alpha_{m,2\ell,s} = \frac{8\pi(3C_A-2T_fN_f)}{3[4(C_A+2C_f)T_fN_f-6C_A^2]} \ . \tag{5.18}$$

At this α, $\beta_{2\ell,s}$ reaches its minimum physical value,

$$(\beta_{2\ell,s})_{min} = -\frac{32\pi(3C_A-2T_fN_f)^3}{27[4(C_A+2C_f)T_fN_f-6C_A^2]^2} \ . \tag{5.19}$$

As in the nonsupersymmetric theory, $\alpha_{m,2\ell,s} = (2/3)\alpha_{IR,2\ell,s}$.

The derivative $d\beta_{2\ell,s}/d\alpha$ evaluated at $\alpha=\alpha_{IR,2\ell,s}$ is given by the analogue of Eq. (3.5), namely

$$\beta'_{IR,2\ell,s} = \frac{2(3C_A-2T_fN_f)^2}{4(C_A+2C_f)T_fN_f-6C_A^2} \ , \tag{5.20}$$

which is positive for $N_f \in I$.

At the three-loop level, $\alpha_{m,3\ell,s}$, $(\beta_{3\ell})_{min}$, and $\beta'_{IR,3\ell,s}$ are given by Eqs. (3.9), (3.12), and (3.16) with the replacements $b_\ell \to b_{\ell,s}$.

In Figs. 3 and 4 we show plots of the two-loop and three-loop β functions for this supersymmetric gauge theory with chiral superfields in the fundamental representation and with the illustrative values $N_c=2$, $N_f=5$ and $N_c=3$, $N_f=7$, respectively. The three-loop β functions are calculated in the $\overline{DR}$ scheme.

VI. CONCLUSIONS

In this paper we have studied some higher-loop structural properties of the β function in an asymptotically free vectorial gauge theory, focusing on the case where the theory has an IR zero in the β function. These structural properties include the value of α where β reaches a minimum (i.e., a maximal magnitude, since $\beta \le 0$ for $\alpha \in I_\alpha$), the value of β at this minimum, and the derivative $d\beta/d\alpha$ at the IR zero, calculated to the n-loop order. We have given results up to four loops in a non-supersymmetric gauge theory and up to three loops in a gauge theory with $\mathcal{N}=1$ supersymmetry. In an asymptotically free theory with an exact or approximate infrared zero in the β function, these structural quantities provide further information about the running of α as a function of the reference scale, μ. The derivative of β at α_{IR} is also of interest because it enters into estimates of the dilaton mass in a quasiconformal gauge theory. A general inequality was proved concerning how the shift in the IR zero of β as one goes from the n-loop to the $(n+1)$-loop order depends on the sign of b_{n+1}. For schemes which have $b_3 < 0$ for $N_f \in I$ and which thus are guaranteed to preserve the existence of the IR zero in the (scheme-independent) $\beta_{2\ell}$ at the three-loop level, we have proved that $\alpha_{IR,3\ell} < \alpha_{IR,2\ell}$, $\alpha_{m,3\ell} < \alpha_{m,2\ell}$, $|(\beta_{3\ell})_{min}| < |(\beta_{2\ell})_{min}|$, and $\beta'_{IR,3\ell} < \beta'_{IR,2\ell}$. Our results further elucidate the ultraviolet to infrared evolution of an asymptotically free vectorial gauge theory with

fermions.

Acknowledgments: I would like to thank T. Ryttov for collaboration on the earlier works [11, 13, 14], and T. Appelquist and the theory group at Yale for useful discussions and warm hospitality during the sabbatical period when this work was done. This research was partially supported by the grant NSF-PHY-09-69739.

VII. APPENDIX A

A. β Function Coefficients

For a vectorial gauge theory with gauge group G and N_f fermions in the representation R, the coefficients b_1 and b_2 in the β function are [1]

$$b_1 = \frac{1}{3}(11C_A - 4T_f N_f) \tag{7.1}$$

and [2, 22]

$$b_2 = \frac{1}{3}\left[34C_A^2 - 4(5C_A + 3C_f)T_f N_f\right] \ . \tag{7.2}$$

In the $\overline{MS}$ scheme [3]

$$b_3 = \frac{2857}{54}C_A^3 + +T_f N_f\left[2C_f^2 - \frac{205}{9}C_A C_f - \frac{1415}{27}C_A^2\right] + (T_f N_f)^2\left[\frac{44}{9}C_f + \frac{158}{27}C_A\right] \ . \tag{7.3}$$

We have also used the four-loop coefficient, b_4, calculated in the $\overline{MS}$ scheme in [4], for our calculations. This coefficient b_4 is a cubic polynomial in N_f.

B. Coefficients for γ_m

We list here the c_ℓ for $\ell = 1,\ 2,\ 3$:

$$c_1 = 6C_f \tag{7.4}$$

$$c_2 = 2C_f\left[\frac{3}{2}C_f + \frac{97}{6}C_A - \frac{10}{3}T_f N_f\right] \ . \tag{7.5}$$

$$c_3 = 2C_f\left[\frac{129}{2}C_f^2 - \frac{129}{4}C_f C_A + \frac{11413}{108}C_A^2 + C_f(T_f N_f)\Big(-46 + 48\zeta(3)\Big) - C_A(T_f N_f)\Big(\frac{556}{27} + 48\zeta(3)\Big) - \frac{140}{27}(T_f N_f)^2\right] , \tag{7.6}$$

where $\zeta(s) = \sum_{n=1}^\infty n^{-s}$ is the Riemann zeta function, and $\zeta(3) = 1.2020569....$ We have also used the four-loop coefficient c_4, calculated in the $\overline{MS}$ scheme in [7], for our calculations.

VIII. APPENDIX B

Here we give some illustrative explicit $\beta_{4\ell}$ functions for various values of N_c and N_f. The three-loop and four-loop coefficients are calculated in the $\overline{MS}$ scheme. These are written in the form

$$\beta_{4\ell} = -2\bar{b}_1\alpha^2\left[1 + \sum_{\ell=2}^{4}\Big(\frac{\bar{b}_\ell}{\bar{b}_1}\Big)\alpha^{\ell-1}\right] \tag{8.1}$$

and are listed both analytically and numerically (to the indicated floating-point accuracy).

$$N_c = 2,\ N_f = 8: \quad \beta_{4\ell} = -\frac{\alpha^2}{\pi}\left[\ 1 - \frac{5}{2}\Big(\frac{\alpha}{\pi}\Big) - \frac{603}{64}\Big(\frac{\alpha}{\pi}\Big)^2 + \Big(\frac{-136859 + 198528\zeta(3)}{9216}\Big)\Big(\frac{\alpha}{\pi}\Big)^3\ \right]$$
$$= -0.3183\alpha^2\Big(1 - 0.7958\alpha - 0.9546\alpha^2 + 0.3562\alpha^3\Big) \tag{8.2}$$

$$N_c = 2,\ N_f = 9: \quad \beta_{4\ell} = -\frac{2\alpha^2}{3\pi}\left[\ 1 - \frac{169}{32}\Big(\frac{\alpha}{\pi}\Big) - \frac{154445}{9216}\Big(\frac{\alpha}{\pi}\Big)^2 + \Big(\frac{-22506041 + 50531904\zeta(3)}{1327104}\Big)\Big(\frac{\alpha}{\pi}\Big)^3\ \right]$$

$$= -0.2122\alpha^2\Big(1 - 1.6811\alpha - 1.6980\alpha^2 + 0.9292\alpha^3\Big) \tag{8.3}$$

$$N_c = 3,\ N_f = 10: \quad \beta_{4\ell} = -\frac{13\alpha^2}{6\pi}\left[1 - \frac{37}{26}\Big(\frac{\alpha}{\pi}\Big) - \frac{41351}{3744}\Big(\frac{\alpha}{\pi}\Big)^2 + \Big(\frac{-13418011 + 13331592\zeta(3)}{404352}\Big)\Big(\frac{\alpha}{\pi}\Big)^3\right]$$
$$= -0.6897\alpha^2\Big(1 - 0.4530\alpha - 1.1191\alpha^2 + 0.2080\alpha^3\Big) \tag{8.4}$$

$$N_c = 3,\ N_f = 12: \quad \beta_{4\ell} = -\frac{3\alpha^2}{2\pi}\left[1 - \frac{25}{6}\Big(\frac{\alpha}{\pi}\Big) - \frac{6361}{288}\Big(\frac{\alpha}{\pi}\Big)^2 + \Big(\frac{-140881 + 219192\zeta(3)}{3456}\Big)\Big(\frac{\alpha}{\pi}\Big)^3\right]$$
$$= -0.4775\alpha^2\Big(1 - 1.3263\alpha - 2.2379\alpha^2 + 1.1441\alpha^3\Big)\ . \tag{8.5}$$

IX. APPENDIX C

Consider the polynomial of degree m in z, $P_m(z) = \sum_{s=0}^{m} \kappa_s z^s$. As discussed in Section II, information on the nature of the roots of the equation $P_m(z) = 0$ is given by the discriminant Δ_m defined in Eq. (2.9). Since Δ_m is a symmetric function of the roots (being proportional to the square of the Vandermonde polynomial of these roots), the theorem on symmetric functions [26] implies that Δ_m can be expressed as a polynomial in the coefficients κ_s, $s = 0, ..., m$. We indicate this in the notation $\Delta_m = \Delta_m(\kappa_0, ..., \kappa_m)$. The discriminant Δ_m is most conveniently calculated in terms of the Sylvester matrix of $P(z)$ and $dP(z)/dz$, equivalent to the resultant matrix, denoted $S_{P,P'}$, of dimension $(2m-1)\times(2m-1)$:

$$\Delta_m = (-1)^{m(m-1)/2}\kappa_m^{-1}\det(S_{P,P'})\ . \tag{9.1}$$

Since we will use Δ_m for $m = 2$ and $m = 3$, we list the explicit expressions here:

$$\Delta_2(\kappa_0, \kappa_1, \kappa_2) = \kappa_1^2 - 4\kappa_0\kappa_2\ . \tag{9.2}$$

For $m = 3$,

$$S_{P_3,P_3'} = \begin{pmatrix} \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0 & 0 \\ 0 & \kappa_3 & \kappa_2 & \kappa_1 & \kappa_0 \\ 3\kappa_3 & 2\kappa_2 & \kappa_1 & 0 & 0 \\ 0 & 3\kappa_3 & 2\kappa_2 & \kappa_1 & 0 \\ 0 & 0 & 3\kappa_2 & 2\kappa_2 & \kappa_1 \end{pmatrix} \tag{9.3}$$

so that

$$\Delta_3(\kappa_0, \kappa_1, \kappa_2, \kappa_3) = (\kappa_1\kappa_2)^2 - 27(\kappa_0\kappa_3)^2 - 4(\kappa_0\kappa_2^3 + \kappa_3\kappa_1^3)$$
$$+\ 18\kappa_0\kappa_1\kappa_2\kappa_3\ . \tag{9.4}$$

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[23] Here and below, when an expression is given for N_f that formally evaluates to a non-integral real value ν, it is understood implicitly that one infers an appropriate integral value of N_f from it, either the greatest integer less than ν, or the smallest integer greater than ν, or the integer nearest to ν, depending on the context. Parenthetically, we note that although our analysis applies to theories with N_f Dirac fermions, in the special case of a fermion in the adjoint representation, one may formally use the value $N_f = 1/2$ to describe a theory with a Majorana fermion.

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TABLE I: Value of $\alpha_{m,n\ell}$ at the n-loop level with $n = 2,\ 3,\ 4$ for an SU(N_c) gauge theory with N_f fermions in the fundamental representation, with $N_f \in I$. As discussed in the text, $\alpha_{m,n\ell}$ is the value at which the n-loop β function takes on its minimum value in the interval $0 \le \alpha \le \alpha_{IR,n\ell}$. Results are given for the illustrative values $N_c = 2,\ 3,\ 4$. For comparison, we also list the IR zeros of β calculated at n-loop level, $\alpha_{IR,n\ell}$, for $n = 2,\ 3,\ 4$, from Ref. [11]. For this and other tables, quantities evaluated at the $n = 3$ and $n = 4$ loop level are calculated in the $\overline{MS}$ scheme.

N_c	N_f	$\alpha_{m,2\ell}$	$\alpha_{m,3\ell}$	$\alpha_{m,4\ell}$	$\alpha_{IR,2\ell}$	$\alpha_{IR,3\ell}$	$\alpha_{IR,4\ell}$
2	7	1.89	0.735	0.823	2.83	1.05	1.21
2	8	0.838	0.476	0.515	1.26	0.688	0.760
2	9	0.397	0.286	0.300	0.595	0.418	0.444
2	10	0.154	0.133	0.135	0.231	0.196	0.200
3	10	1.47	0.534	0.563	2.21	0.764	0.815
3	11	0.823	0.402	0.429	1.23	0.579	0.626
3	12	0.503	0.300	0.320	0.754	0.435	0.470
3	13	0.312	0.217	0.228	0.468	0.317	0.337
3	14	0.185	0.146	0.151	0.278	0.215	0.224
3	15	0.0952	0.0834	0.0846	0.143	0.123	0.126
3	16	0.0277	0.0416	0.0267	0.0416	0.0397	0.0398
4	13	1.23	0.422	0.436	1.85	0.604	0.628
4	14	0.773	0.340	0.359	1.16	0.489	0.521
4	15	0.522	0.275	0.293	0.783	0.397	0.428
4	16	0.364	0.221	0.235	0.546	0.320	0.345
4	17	0.256	0.174	0.184	0.384	0.254	0.271
4	18	0.177	0.133	0.138	0.266	0.194	0.205
4	19	0.117	0.0954	0.0981	0.175	0.140	0.145
4	20	0.0697	0.0613	0.0621	0.105	0.0907	0.0924
4	21	0.0315	0.0472	0.0297	0.0472	0.044	0.0444

TABLE II: Values of the discriminants $\Delta_2(\bar{b}_1, \bar{b}_2, \bar{b}_3)$ and $\Delta_3(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$ (see Eqs. (2.9) and (2.10)) for the three-loop and four-loop IR zero equations, with $\bar{b}_3$ and $\bar{b}_4$ calculated in the $\overline{MS}$ scheme. Results are given for the illustrative values $N_c = 2,\ 3,\ 4$. Notation ae-n means $a \times 10^{-n}$.

N_c	N_f	$\Delta_2(\bar{b}_1, \bar{b}_2, \bar{b}_3)$	$\Delta_3(\bar{b}_1, \bar{b}_2, \bar{b}_3, \bar{b}_4)$
2	7	0.107	0.151e-2
2	8	0.113	0.399e-2
2	9	0.108	0.885e-2
2	10	0.0963	1.68e-2
3	10	0.557	0.0943
3	11	0.596	0.170
3	12	0.610	0.293
3	13	0.603	0.493
3	14	0.577	0.803
3	15	0.537	1.221
3	16	0.489	1.676
4	13	1.75	1.53
4	14	1.87	2.45
4	15	1.95	3.74
4	16	1.97	5.60
4	17	1.96	8.32
4	18	1.92	12.18
4	19	1.85	17.36
4	20	1.75	23.71
4	21	1.64	30.59

TABLE III: Minimum value of the n-loop β function, $\beta_{n\ell}$, denoted $(\beta_{n\ell})_{min}$, in the interval $0 \le \alpha \le \alpha_{IR,n\ell}$ relevant for the UV to IR evolution, calculated to $n = 2,\ 3,\ 4$ loop order for an SU(N_c) theory with N_f fermions in the fundamental representation, with $N_f \in I$. Values are given for $N_c = 2,\ 3,\ 4$. Notation ae-n means $a \times 10^{-n}$.

N_c	N_f	$(\beta_{2\ell})_{min}$	$(\beta_{3\ell})_{min}$	$(\beta_{4\ell})_{min}$
2	7	−0.504	−0.998e-1	−0.117
2	8	−0.745e-1	−0.292e-1	−0.326e-1
2	9	−1.11e-2	−0.660e-2	−0.703e-2
2	10	−0.836e-3	−0.666e-3	−0.680e-3
3	10	−0.498	−0.863e-1	−0.934e-1
3	11	−1.32e-1	−0.394e-1	−0.432e-1
3	12	−0.4025e-1	−1.72e-2	−1.88e-2
3	13	−1.20e-2	−0.672e-2	−0.719e-2
3	14	−0.304e-2	−0.209e-2	−0.218e-2
3	15	−0.481e-3	−0.392e-3	−0.399e-3
3	16	−1.36e-5	−1.28e-5	−1.28e-5
4	13	−0.484	−0.752e-1	−0.790e-1
4	14	−0.169	−0.419e-1	−0.452e-1
4	15	−0.674e-1	−0.232e-1	−0.252e-1
4	16	−0.2815e-1	−1.24e-2	−1.35e-2
4	17	−1.16e-2	−0.6215e-2	−0.667e-2
4	18	−0.444e-2	−0.280e-2	−0.296e-2
4	19	−1.45e-3	−1.055e-3	−1.09e-3
4	20	−0.343e-3	−0.282e-3	−0.287e-3
4	21	−0.350e-4	−0.319e-4	−0.321e-4

TABLE IV: Value of $d\beta_{n\ell}/d\alpha$ at $n = 2,\ 3,\ 4$ loop order for an SU(N_c) theory with N_f fermions in the fundamental representation, with $N_f \in I$, evaluated at the IR zero calculated to this order, $\alpha_{IR,n\ell}$. We denote this here as $\beta'_{IR,n\ell}$.

N_c	N_f	$\beta'_{IR,2\ell}$	$\beta'_{IR,3\ell}$	$\beta'_{IR,4\ell}$
2	7	1.20	0.728	0.677
2	8	0.400	0.318	0.300
2	9	0.126	0.115	0.110
2	10	0.0245	0.0239	0.0235
3	10	1.52	0.872	0.853
3	11	0.720	0.517	0.498
3	12	0.360	0.2955	0.282
3	13	0.174	0.156	0.149
3	14	0.0737	0.0699	0.0678
3	15	0.0227	0.0223	0.0220
3	16	0.00221	0.00220	0.00220
4	13	1.77	0.965	0.955
4	14	0.984	0.655	0.639
4	15	0.581	0.440	0.424
4	16	0.348	0.288	0.276
4	17	0.204	0.180	0.1725
4	18	0.113	0.105	0.101
4	19	0.0558	0.0536	0.0522
4	20	0.0222	0.0218	0.0.215
4	21	0.00501	0.00499	0.00496